

# HIGHEST WEIGHT VECTORS OF MIXED TENSOR PRODUCTS OF GENERAL LINEAR LIE SUPERALGEBRAS

HEBING RUI AND YUCAI SU

**ABSTRACT.** In this paper, a notion of cyclotomic (or level  $k$ ) walled Brauer algebras  $\mathcal{B}_{k,r,t}$  is introduced for arbitrary positive integer  $k$ . It is proven that  $\mathcal{B}_{k,r,t}$  is free over a commutative ring with rank  $k^{r+t}(r+t)!$  if and only if it is admissible. Using super Schur-Weyl duality between general linear Lie superalgebras  $\mathfrak{gl}_{m|n}$  and  $\mathcal{B}_{2,r,t}$ , we give a classification of highest weight vectors of  $\mathfrak{gl}_{m|n}$ -modules  $M_{pq}^{rt}$ , the tensor products of Kac-modules with mixed tensor products of the natural module and its dual. This enables us to establish an explicit relationship between  $\mathfrak{gl}_{m|n}$ -Kac-modules and right cell (or standard)  $\mathcal{B}_{2,r,t}$ -modules over  $\mathbb{C}$ . Further, we find an explicit relationship between indecomposable tilting  $\mathfrak{gl}_{m|n}$ -modules appearing in  $M_{pq}^{rt}$ , and principal indecomposable right  $\mathcal{B}_{2,r,t}$ -modules via the notion of Kleshchev bipartitions. As an application, decomposition numbers of  $\mathcal{B}_{2,r,t}$  arising from super Schur-Weyl duality are determined.

## 1. INTRODUCTION

Motivated by Brundan-Stroppel's work on higher super Schur-Weyl duality in [5], we introduced affine walled Brauer algebras  $\mathcal{B}_{r,t}^{\text{aff}}$  in [19] so as to establish higher super Schur-Weyl duality on the tensor product  $M_{pq}^{rt}$  of a Kac-module with a mixed tensor product of the natural module and its dual for general linear Lie superalgebra  $\mathfrak{gl}_{m|n}$  over  $\mathbb{C}$  under the assumption  $r+t \leq \min\{m, n\}$ .<sup>1</sup> One of purposes of this paper is to generalize super Schur-Weyl duality to the case  $r+t > \min\{m, n\}$ . For this aim, we need to establish a bijective map from a level two walled Brauer algebra  $\mathcal{B}_{2,r,t}$  appearing in [19] to a level two degenerate Hecke algebra  $\mathcal{H}_{2,r+t}$ . This can be done by showing that the dimension of  $\mathcal{B}_{2,r,t}$  is  $2^{r+t}(r+t)!$  over  $\mathbb{C}$ . We consider this problem in a general setting by introducing a cyclotomic (or level  $k$ ) walled Brauer algebra  $\mathcal{B}_{k,r,t}$  for arbitrary  $k \in \mathbb{Z}^{>0}$ . By employing a totally new method, which is independent of seminormal forms of  $\mathcal{B}_{k,r,t}$ , we prove that  $\mathcal{B}_{k,r,t}$  is free over a commutative ring  $R$  with rank  $k^{r+t}(r+t)!$  if and only if it is admissible in the sense of Definition 2.3. It is expected that  $\mathcal{B}_{k,r,t}$  can be used to study the problem on a classification of finite dimensional simple  $\mathcal{B}_{r,t}^{\text{aff}}$ -modules over an algebraically closed field. Details will be given elsewhere.

The establishment of the higher super Schur-Weyl duality [19] enables us to use the representation theory of  $\mathcal{B}_{2,r,t}$  to classify highest weight vectors of  $M_{pq}^{rt}$  (at this point, we would like to mention that purely on the Lie superalgebra side, it seems to be hard to construct highest weight vectors of a given module, which is an interesting problem on its own right). On the other hand, a classification of highest weight vectors of  $M_{pq}^{rt}$  also enables us to relate the category of finite dimensional  $\mathfrak{gl}_{m|n}$ -modules with that of  $\mathcal{B}_{2,r,t}$ , which in turn gives us

Supported by NSFC (grant no. 11025104, 11371278), Shanghai Municipal Science and Technology Commission 11XD1402200, 12XD1405000, and Fundamental Research Funds for the Central Universities of China.

<sup>1</sup>After we finished [19], Professor Stroppel informed us that Sartori defined affine walled algebras via affine walled Brauer category, independently in [20].

an efficient way to calculate decomposition numbers of  $\mathcal{B}_{2,r,t}$  (cf. [18] for quantum walled Brauer algebras). This is the main motivation of this paper. We explain some details below.

It is proven in [19] that  $\text{End}_{U(\mathfrak{gl}_{m|n})}(M_{pq}^{rt}) \cong \mathcal{B}_{2,r,t}$  if  $r + t \leq \min\{m, n\}$ . Since there is a bijection between the dominant weights of  $M_{pq}^{rt}$  and the poset  $\Lambda_{2,r,t}$  in (3.12), and since  $\mathcal{B}_{2,r,t}$  is a weakly cellular algebra over  $\Lambda_{2,r,t}$  in the sense of [11], it is very natural to ask the following problem: whether a  $\mathbb{C}$ -space of  $\mathfrak{gl}_{m|n}$ -highest weight vectors of  $M_{pq}^{rt}$  with a fixed highest weight is isomorphic to a cell (or standard) module of  $\mathcal{B}_{2,r,t}$ .

We give an affirmative answer to the problem. In sharp contrast to the Lie algebra case, due to the existence of the parity of  $\mathfrak{gl}_{m|n}$ , the known weakly cellular basis of  $\mathcal{B}_{2,r,t}$  in [19] can not be directly used to establish a relationship between  $\mathfrak{gl}_{m|n}$ -highest weight vectors of  $M_{pq}^{rt}$  and right cell modules of  $\mathcal{B}_{2,r,t}$ . One has to find new cellular bases of level two Hecke algebra  $\mathcal{H}_{2,r}$  which are different from that in [3]. These new cellular bases of  $\mathcal{H}_{2,r}$ , which relate both trivial and signed representations of symmetric groups, are used to construct a new weakly cellular basis of  $\mathcal{B}_{2,r,t}$ . Motivated by explicit descriptions of bases of right cell modules for  $\mathcal{B}_{2,r,t}$ , we construct and classify  $\mathfrak{gl}_{m|n}$ -highest weight vectors of  $M_{pq}^{rt}$ . This leads to a  $\mathcal{B}_{2,r,t}$ -module isomorphism between each  $\mathbb{C}$ -space of  $\mathfrak{gl}_{m|n}$ -highest weight vectors of  $M_{pq}^{rt}$  with a fixed highest weight and the corresponding cell module of  $\mathcal{B}_{2,r,t}$ . Based on the above, we are able to construct a suitable exact functor sending  $\mathfrak{gl}_{m|n}$ -Kac-modules to right cell modules of  $\mathcal{B}_{2,r,t}$ . This functor also sends an indecomposable tilting module appearing in  $M_{pq}^{rt}$  to a principal indecomposable right  $\mathcal{B}_{2,r,t}$ -module indexed by a pair of so-called Kleshchev bipartitions in the sense of (3.15). It gives us an efficient way to calculate decomposition numbers of  $\mathcal{B}_{2,r,t}$  via Brundan-Stroppel's result [5] on the multiplicity of a Kac-module in an indecomposable tilting module appearing in  $M_{pq}^{rt}$ .

We organize the paper as follows. In section 2, after recalling the definition of  $\mathcal{B}_{r,t}^{\text{aff}}$  over a commutative ring  $R$ , we introduce cyclotomic walled Brauer algebras  $\mathcal{B}_{k,r,t} := \mathcal{B}_{r,t}^{\text{aff}}/I$  for arbitrary  $k \in \mathbb{Z}^{>0}$ , where  $I$  is the two-sided ideal of  $\mathcal{B}_{r,t}^{\text{aff}}$  generated by two cyclotomic polynomials  $\mathbf{f}(x_1)$  and  $\mathbf{g}(\bar{x}_1)$  of degree  $k$ , which satisfy (2.5)–(2.7). When  $\mathcal{B}_{r,t}^{\text{aff}}$  is admissible in the sense of Definition 2.3, we describe explicitly an  $R$ -basis of  $I$ . This enables us to prove that  $\mathcal{B}_{k,r,t}$  is free over  $R$  with rank  $k^{r+t}(r+t)!$  if and only if it is admissible. In section 3, we construct cellular bases of  $\mathcal{H}_{2,r}$  and use them to construct a weakly cellular basis of  $\mathcal{B}_{2,r,t}$ . In section 4, higher super Schur-Weyl dualities in [19] are generalized to the case  $r + t > \min\{m, n\}$ . In sections 5–6, we classify highest weight vectors of  $M_{pq}^{r0}$  and  $M_{pq}^{rt}$ . Based on this, we establish an explicit relationship between indecomposable tilting (resp. Kac) modules for  $\mathfrak{gl}_{m|n}$  and principal indecomposable (resp. cell) right  $\mathcal{B}_{2,r,t}$ -modules via a suitable exact functor. This gives us an efficient way to calculate decomposition numbers of  $\mathcal{B}_{2,r,t}$  arising from the super Schur-Weyl duality in [19].

## 2. AFFINE WALLED BRAUER ALGEBRAS AND ITS CYCLOTOMIC QUOTIENTS

Throughout, we assume that  $R$  is a commutative ring containing  $\mathbf{\Omega} = \{\omega_a \mid a \in \mathbb{N}\}$  and identity 1. In this section, we introduce a level  $k$  walled Brauer algebra  $\mathcal{B}_{k,r,t}$  and prove that

$\mathcal{B}_{k,r,t}$  is free over  $R$  with rank  $k^{r+t}(r+t)!$  if and only if  $\mathcal{B}_{k,r,t}$  is admissible in the sense of Definition 2.3. First, we briefly recall the definition of walled Brauer algebras.

Fix  $r, t \in \mathbb{Z}^{>0}$ . A *walled  $(r, t)$ -Brauer diagram* (or simply, a *walled Brauer diagram*) is a diagram with  $(r+t)$  vertices on top and bottom rows, and vertices on both rows are labeled from left to right by  $r, \dots, 2, 1, \bar{1}, \bar{2}, \dots, \bar{t}$ , such that every  $i \in \{r, \dots, 2, 1\}$  (resp.,  $\bar{i} \in \{\bar{1}, \bar{2}, \dots, \bar{t}\}$ ) on each row is connected to a unique  $\bar{j}$  (resp.,  $j$ ) on the same row or a unique  $j$  (resp.,  $\bar{j}$ ) on the other row. Thus there are four types of pairs  $[i, j]$ ,  $[i, \bar{j}]$ ,  $[\bar{i}, j]$  and  $[\bar{i}, \bar{j}]$ . The pairs  $[i, j]$  and  $[\bar{i}, \bar{j}]$  are *vertical edges*, and  $[\bar{i}, j]$  and  $[i, \bar{j}]$  are *horizontal edges*.

The product of two walled Brauer diagrams  $D_1$  and  $D_2$  can be defined via concatenation. Putting  $D_1$  above  $D_2$  and connecting each vertex on the bottom row of  $D_1$  to the corresponding vertex on the top row of  $D_2$  yields a diagram  $D_1 \circ D_2$ , called the *concatenation* of  $D_1$  and  $D_2$ . Removing all circles of  $D_1 \circ D_2$  yields a unique walled Brauer diagram, denoted  $D_3$ . Let  $n$  be the number of circles appearing in  $D_1 \circ D_2$ . Then the *product*  $D_1 D_2$  is defined to be  $\omega_0^n D_3$ , where  $\omega_0$  is a fixed element in  $R$ . The *walled Brauer algebra* [16, 23, 17]  $\mathcal{B}_{r,t} := \mathcal{B}_{r,t}(\omega_0)$  with defining parameter  $\omega_0$  is the associative  $R$ -algebra spanned by all walled Brauer diagrams with product defined in this way.

Let  $\mathfrak{S}_r$  (resp.  $\tilde{\mathfrak{S}}_t$ ) be the symmetric group in  $r$  (resp.  $t$ ) letters  $r, \dots, 2, 1$  (resp.  $\bar{1}, \bar{2}, \dots, \bar{t}$ ). It is known that  $\mathcal{B}_{r,t}$  contains two subalgebras which are isomorphic to the group algebras of  $\mathfrak{S}_r$  and  $\tilde{\mathfrak{S}}_t$ , respectively. More explicitly, the walled Brauer diagram  $s_i$  whose edges are of forms  $[k, k]$  and  $[\bar{k}, \bar{k}]$  except two vertical edges  $[i, i+1]$  and  $[i+1, i]$  can be identified with the basic transposition  $(i, i+1) \in \mathfrak{S}_r$ , which switches  $i$  and  $i+1$  and fixes others. Similarly, there is a walled Brauer diagram  $\bar{s}_j$  corresponding to  $(\bar{j}, \overline{j+1}) \in \tilde{\mathfrak{S}}_t$ . Let  $e_1$  be the walled Brauer diagram whose edges are of forms  $[k, k]$  and  $[\bar{k}, \bar{k}]$  except two horizontal edges  $[1, \bar{1}]$  on the top and bottom rows. Then  $\mathcal{B}_{r,t}$  is the  $R$ -algebra [17] generated by  $e_1, s_i, \bar{s}_j$  for  $1 \leq i \leq r-1, 1 \leq j \leq t-1$  such that  $s_i$ 's,  $\bar{s}_j$ 's are distinguished generators of  $\mathfrak{S}_r \times \tilde{\mathfrak{S}}_t$  and

$$\begin{aligned} e_1^2 &= \omega_0 e_1, & e_1 s_1 e_1 &= e_1 = e_1 \bar{s}_1 e_1, & s_i e_1 &= e_1 s_i, & \bar{s}_j e_1 &= e_1 \bar{s}_j \quad (i, j \neq 1), \\ e_1 s_1 \bar{s}_1 e_1 s_1 &= e_1 s_1 \bar{s}_1 e_1 \bar{s}_1, & s_1 e_1 s_1 \bar{s}_1 e_1 &= \bar{s}_1 e_1 s_1 \bar{s}_1 e_1. \end{aligned} \quad (2.1)$$

Let  $\mathcal{H}_n^{\text{aff}}$  be the *degenerate affine Hecke algebra* [10]. As a free  $R$ -module, it is the tensor product  $R[y_1, y_2, \dots, y_n] \otimes R\mathfrak{S}_n$  of a polynomial algebra with the group algebra of  $\mathfrak{S}_n$ . The multiplication is defined so that  $R[y_1, y_2, \dots, y_n] \equiv R[y_1, y_2, \dots, y_n] \otimes 1$  and  $R\mathfrak{S}_n \equiv 1 \otimes R\mathfrak{S}_n$  are subalgebras and  $s_i y_j = y_j s_i$  if  $j \neq i, i+1$  and  $s_i y_i = y_{i+1} s_i - 1, 1 \leq i \leq n-1$ .

Recall that  $R$  contains 1 and  $\Omega = \{\omega_a \in R \mid a \in \mathbb{N}\}$ . The *affine walled Brauer algebra*  $\mathcal{B}_{r,t}^{\text{aff}}(\Omega)$  (which is  $\widehat{\mathcal{B}_{r,t}}$  in [19, §4]) with respect to the defining parameters  $\omega_a$ 's have been defined via generators and 26 defining relations [19, Definition 2.7]. It follows from [19, Theorem 4.15] that  $\mathcal{B}_{r,t}^{\text{aff}}(\Omega)$  can be also defined in a simpler way as follows: it is an associative  $R$ -algebra generated by  $e_1, x_1, \bar{x}_1, s_i, \bar{s}_j$  for  $1 \leq i \leq r-1, 1 \leq j \leq t-1$ , such that  $e_1, s_i$ 's,  $\bar{s}_j$ 's are generators of  $\mathcal{B}_{r,t}$  with defining parameter  $\omega_0$ , and as a free  $R$ -module,

$$\mathcal{B}_{r,t}^{\text{aff}}(\Omega) = R[\mathbf{x}_r] \otimes \mathcal{B}_{r,t} \otimes R[\bar{\mathbf{x}}_t],$$

the tensor product of the walled Brauer algebra  $\mathcal{B}_{r,t}$  with two polynomial algebras

$$R[\mathbf{x}_r] := R[x_1, x_2, \dots, x_r], \quad \text{and} \quad R[\bar{\mathbf{x}}_t] := R[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_t].$$

Multiplication is defined as  $R[\mathbf{x}_r] \equiv R[\mathbf{x}_r] \otimes 1 \otimes 1$ ,  $R[\bar{\mathbf{x}}_r] \equiv 1 \otimes 1 \otimes R[\bar{\mathbf{x}}_r]$ , and  $\mathcal{B}_{r,t} \equiv 1 \otimes \mathcal{B}_{r,t} \otimes 1$ , and  $R[\mathbf{x}_r] \otimes R\mathfrak{S}_r \otimes 1 \cong \mathcal{H}_r^{\text{aff}} \otimes 1$ ,  $1 \otimes R\bar{\mathfrak{S}}_t \otimes R[\bar{\mathbf{x}}_r] \cong 1 \otimes \mathcal{H}_t^{\text{aff}}$  and

$$e_1(x_1 + \bar{x}_1) = (x_1 + \bar{x}_1)e_1 = 0, \quad s_1 e_1 s_1 x_1 = x_1 s_1 e_1 s_1, \quad \bar{s}_1 e_1 \bar{s}_1 \bar{x}_1 = \bar{x}_1 \bar{s}_1 e_1 \bar{s}_1, \quad (2.2)$$

$$s_i \bar{x}_1 = \bar{x}_1 s_i, \quad \bar{s}_i x_1 = x_1 \bar{s}_i, \quad x_1(e_1 + \bar{x}_1) = (e_1 + \bar{x}_1)x_1, \quad (2.3)$$

$$e_1 x_1^k e_1 = \omega_k e_1, \quad e_1 \bar{x}_1^k e_1 = \bar{\omega}_k e_1, \quad \forall k \in \mathbb{Z}^{\geq 0}, \quad (2.4)$$

where  $\bar{\omega}_a$ 's are determined by [19, Corollary 4.3]. If  $\bar{\omega}_a$ 's do not satisfy [19, Corollary 4.3], then  $e_1 = 0$  and  $\mathcal{B}_{r,t}^{\text{aff}}(\Omega)$  turns out to be  $\mathcal{H}_r^{\text{aff}} \otimes \mathcal{H}_t^{\text{aff}}$  if  $R$  is a field.

We remark that the isomorphism  $R[\mathbf{x}_r] \otimes R\mathfrak{S}_r \otimes 1 \cong \mathcal{H}_r^{\text{aff}} \otimes 1$  sends  $s_i$ 's (resp.  $x_1$ ) to  $s_i$ 's (resp.  $-y_1$ ), and the isomorphism  $1 \otimes R\bar{\mathfrak{S}}_t \otimes R[\bar{\mathbf{x}}_r] \cong 1 \otimes \mathcal{H}_t^{\text{aff}}$  sends  $\bar{s}_j$ 's (resp.  $\bar{x}_1$ ) to  $s_j$ 's (resp.  $-y_1$ ). So,  $x_{i+1} = s_i x_i s_i - s_i$  and  $\bar{x}_{j+1} = \bar{s}_j \bar{x}_j \bar{s}_j - \bar{s}_j$  and  $y_{i+1} = s_i y_i s_i + s_i$  if all of them make sense.

For the simplification of notation, we denote  $\mathcal{B}_{r,t}^{\text{aff}}(\Omega)$  by  $\mathcal{B}_{r,t}^{\text{aff}}$ . Fix  $u_1, u_2, \dots, u_k \in R$  for some  $k \in \mathbb{Z}^{>0}$ . Let  $\mathbf{f}(x_1) \in \mathcal{B}_{r,t}^{\text{aff}}$  be such that

$$\mathbf{f}(x_1) = \prod_{i=1}^k (x_1 - u_i). \quad (2.5)$$

By [19, Lemma 4.2] (or using (2.2)–(2.3)), there is a monic polynomial  $\mathbf{g}(\bar{x}_1) \in R[\bar{x}_1]$  with degree  $k$  such that

$$e_1 \mathbf{f}(x_1) = (-1)^k e_1 \mathbf{g}(\bar{x}_1). \quad (2.6)$$

If  $R$  is an algebraically closed field, then there are  $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k \in R$  such that

$$\mathbf{g}(\bar{x}_1) = \prod_{i=1}^k (\bar{x}_1 - \bar{u}_i). \quad (2.7)$$

**Definition 2.1.** Let  $R$  be a commutative ring containing 1,  $\Omega = \{\omega_a \in R \mid a \in \mathbb{N}\}$ , and  $u_i, \bar{u}_i$ ,  $1 \leq i \leq k$ . The cyclotomic (or level  $k$ ) walled Brauer algebra  $\mathcal{B}_{k,r,t}$  is the quotient algebra  $\mathcal{B}_{r,t}^{\text{aff}}/I$ , where  $I$  is the two-sided ideal of  $\mathcal{B}_{r,t}^{\text{aff}}$  generated by  $\mathbf{f}(x_1)$  and  $\mathbf{g}(\bar{x}_1)$  satisfying (2.5)–(2.7).

If  $k = 1$ , then  $\mathcal{B}_{k,r,t}$  is  $\mathcal{B}_{r,t}$  with defining parameter  $\omega_0$ . For some special  $u_i, \bar{u}_i$ ,  $i = 1, 2$ ,  $\mathcal{B}_{2,r,t}$  is the level two walled Brauer algebras arising from super Schur-Weyl duality in [19].

**Lemma 2.2.** Let  $\mathbf{f}(x_1)$  be given in (2.5). Write  $\mathbf{f}(x_1) = x_1^k + \sum_{i=1}^k a_i x_1^{k-i}$ . Then  $e_1$  is an  $R$ -torsion element of  $\mathcal{B}_{k,r,t}$  unless

$$\omega_\ell = -(a_1 \omega_{\ell-1} + \dots + a_k \omega_{\ell-k}) \quad \text{for all } \ell \geq k. \quad (2.8)$$

*Proof.* Let  $b_\ell = \omega_\ell + a_1 \omega_{\ell-1} + \dots + a_k \omega_{\ell-k} \in R$ . By (2.4),  $b_\ell e_1 = e_1 \mathbf{f}(x_1) x_1^{\ell-k} e_1$  in  $\mathcal{B}_{r,t}^{\text{aff}}$  and  $b_\ell e_1 = 0$  in  $\mathcal{B}_{k,r,t}$ . Thus,  $e_1$  is an  $R$ -torsion element if  $b_\ell \neq 0$  for some  $\ell \geq k$ .  $\square$

**Definition 2.3.** The algebras  $\mathcal{B}_{r,t}^{\text{aff}}$  and  $\mathcal{B}_{k,r,t}$  are called *admissible* if (2.8) holds.

**Lemma 2.4.** Assume  $\mathbf{f}(x_1), \mathbf{g}(\bar{x}_1) \in \mathcal{B}_{r,t}^{\text{aff}}$  satisfying (2.5)–(2.7). If  $\mathcal{B}_{r,t}^{\text{aff}}$  is admissible, then

- (1)  $e_1 \mathbf{f}(x_1) x_1^a e_1 = 0$  for all  $a \in \mathbb{N}$ .
- (2)  $e_1 \mathbf{g}(\bar{x}_1) \bar{x}_1^a e_1 = 0$  for all  $a \in \mathbb{N}$ .

*Proof.* (1) trivial since  $\mathcal{B}_{r,t}^{\text{aff}}$  is admissible. It is proven in [19] that there is an  $R$ -linear anti-involution  $\sigma$  on  $\mathcal{B}_{r,t}^{\text{aff}}$ , which fixes all generators of  $\mathcal{B}_{r,t}^{\text{aff}}$ . Applying  $\sigma$  on [19, Lemma 4.2] yields

$$\bar{x}_1^k e_1 = \sum_{i=0}^k a_{k,i} x_1^i e_1, \text{ for some } a_{k,i} \in R.$$

So, (2) follows from (2.6) and (1), immediately.  $\square$

Denote  $s_{i,j} = s_i s_{i+1} \cdots s_{j-1}$  if  $i < j$ , and 1 if  $i = j$ , and  $s_{i-1} s_{i-2} \cdots s_j$  if  $i > j$ . Denote  $\bar{s}_{i,j} \in \bar{\mathfrak{S}}_t$  similarly. Let  $e_{i,j}$  be the walled Brauer diagram such that each vertical edge of  $e_{i,j}$  is of form  $[k, k]$  or  $[\bar{k}, \bar{k}]$  and the horizontal edges on the top and bottom rows of  $e_{i,j}$  are  $[i, \bar{j}]$ . Then

$$e_{i,j} = \bar{s}_{j,1} s_{i,1} e_1 s_{1,i} \bar{s}_{1,j} \text{ for } i, j \text{ with } 1 \leq i \leq r \text{ and } 1 \leq j \leq t. \quad (2.9)$$

For each nonnegative integer  $f \leq \min\{r, t\}$ , let

$$e^f = e_1 e_2 \cdots e_f \text{ for } f > 0 \text{ and } e^0 = 1, \text{ where } e_i = e_{i,i}. \quad (2.10)$$

Set

$$\mathcal{D}_{r,t}^f = \{s_{f,i_f} \bar{s}_{f,j_f} \cdots s_{1,i_1} \bar{s}_{1,j_1} \mid 1 \leq i_1 < \cdots < i_f \leq r, k \leq j_k\}. \quad (2.11)$$

**Definition 2.5.** For  $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$  and  $\beta = (\beta_1, \dots, \beta_t) \in \mathbb{N}^t$ , let  $x^\alpha = \prod_{i=1}^r x_i^{\alpha_i}$ ,  $\bar{x}^\beta = \prod_{j=1}^t \bar{x}_j^{\beta_j}$ . Let  $\mathcal{M}$  be a subset of  $\mathcal{B}_{r,t}^{\text{aff}}$  given by

$$\mathcal{M} = \bigcup_{f=0}^{\min\{m,n\}} \{x^\alpha c^{-1} e^f w d \bar{x}^\beta \mid (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t, c, d \in \mathcal{D}_{r,t}^f, w \in \mathfrak{S}_{r-f} \times \bar{\mathfrak{S}}_{t-f}\}. \quad (2.12)$$

Elements of  $\mathcal{M}$  are called *regular monomials* of  $\mathcal{B}_{r,t}^{\text{aff}}$ .

**Theorem 2.6.** [19, Theorem 4.15] *The affine walled Brauer algebra  $\mathcal{B}_{r,t}^{\text{aff}}$  is free over  $R$  with  $\mathcal{M}$  as its  $R$ -basis.*

We consider  $\mathcal{B}_{r,t}^{\text{aff}}$  as a filtrated  $R$ -algebra as follows. Let

$$\deg s_i = \deg \bar{s}_j = \deg e_1 = 0 \text{ and } \deg x_k = \deg \bar{x}_\ell = 1$$

for all possible  $i, j, k, \ell$ 's. Let  $(\mathcal{B}_{r,t}^{\text{aff}})^{(k)}$  be the  $R$ -submodule spanned by regular monomials with degrees less than or equal to  $k$  for  $k \in \mathbb{Z}^{\geq 0}$ . Then we have the following filtration

$$\mathcal{B}_{r,t}^{\text{aff}} \supset \cdots \supset (\mathcal{B}_{r,t}^{\text{aff}})^{(1)} \supset (\mathcal{B}_{r,t}^{\text{aff}})^{(0)} \supset (\mathcal{B}_{r,t}^{\text{aff}})^{(-1)} = 0. \quad (2.13)$$

Let  $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}}) = \bigoplus_{i \geq 0} (\mathcal{B}_{r,t}^{\text{aff}})^{[i]}$ , where  $(\mathcal{B}_{r,t}^{\text{aff}})^{[i]} = (\mathcal{B}_{r,t}^{\text{aff}})^{(i)} / (\mathcal{B}_{r,t}^{\text{aff}})^{(i-1)}$ . Then  $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}})$  is an associated  $\mathbb{Z}$ -graded algebra. We will use the same symbols to denote elements in  $\text{gr}(\mathcal{B}_{r,t}^{\text{aff}})$ .

**Lemma 2.7.** *Let  $x'_i = s_{i-1} x'_{i-1} s_{i-1}$ , and  $\bar{x}'_j = \bar{s}_{j-1} \bar{x}'_{j-1} \bar{s}_{j-1}$  for  $i, j \in \mathbb{Z}^{\geq 2}$  with  $i \leq r$  and  $j \leq t$ , where  $x'_1 = x_1$ , and  $\bar{x}'_1 = \bar{x}_1$ .*

- (1)  $x_i = x'_i - L_i$ , where  $L_i = \sum_{1 \leq j < i} (j, i)$  and  $(j, i)$  is the transposition in  $\mathfrak{S}_r$  which switches  $j, i$  and fixes others.
- (2)  $\bar{x}_i = \bar{x}'_i - \bar{L}_i$ , where  $\bar{L}_i = \sum_{1 \leq \bar{j} < \bar{i}} (\bar{j}, \bar{i})$  and  $(\bar{j}, \bar{i})$  is the transposition in  $\bar{\mathfrak{S}}_t$  which switches  $\bar{j}, \bar{i}$  and fixes others.
- (3) Any symmetric polynomial of  $L_1, L_2, \dots, L_r$  (resp.  $\bar{L}_1, \bar{L}_2, \dots, \bar{L}_t$ ) is a central element of  $R\mathfrak{S}_r$  (resp.  $R\bar{\mathfrak{S}}_t$ ).

*Proof.* (1)-(2) are trivial and (3) is a well-known result.  $\square$

The elements  $L_i$ 's (resp.  $\bar{L}_j$ 's) are known as Jucys-Murphy elements of  $R\mathfrak{S}_r$  (resp.  $R\tilde{\mathfrak{S}}_t$ ). Note that  $x_i x_j = x_j x_i$  and  $\bar{x}_i \bar{x}_j = \bar{x}_j \bar{x}_i$  for all possible  $i, j$ . However,  $x'_i$  and  $x'_j$  (resp.  $\bar{x}'_i$  and  $\bar{x}'_j$ ) do not commute each other.

Suppose  $0 < f \leq \min\{m, n\}$ . Denote

$$\vec{i} = (i_1, \dots, i_f), \quad \vec{j} = (j_1, \dots, j_f), \quad e_{\vec{i}, \vec{j}} = e_{i_1, j_1} e_{i_2, j_2} \cdots e_{i_f, j_f}, \quad (2.14)$$

where  $i_1, i_2, \dots, i_f$  are distinct numbers in  $\{1, 2, \dots, r\}$ , and  $j_1, j_2, \dots, j_f$  are distinct numbers in  $\{\bar{1}, \bar{2}, \dots, \bar{t}\}$ . Then  $e_{i_k, j_k}$ 's commute each other. If  $f = 0$ , we set  $\vec{i} = \vec{j} = \emptyset$  and  $e_{\vec{i}, \vec{j}} = 1$ .

We always assume that  $\mathfrak{S}_r$  (resp.  $\tilde{\mathfrak{S}}_t$ ) acts on the right of  $\{r, \dots, 2, 1\}$  (resp.  $\{\bar{1}, \bar{2}, \dots, \bar{t}\}$ ).

**Lemma 2.8.** *Suppose  $a \in \mathbb{Z}^{>0}$ ,  $1 \leq i, \ell \leq r$  and  $1 \leq j \leq t$ .*

- (1) *If  $w \in \mathfrak{S}_r$ , then  $w\mathbf{f}(x'_i)w^{-1} = \mathbf{f}(x'_{(i)w^{-1}})$ .*
- (2) *If  $w \in \tilde{\mathfrak{S}}_t$ , then  $w\mathbf{g}(\bar{x}'_j)w^{-1} = \mathbf{g}(\bar{x}'_{(j)w^{-1}})$ .*
- (3)  *$x'_i \mathbf{f}(x'_\ell) = \mathbf{f}(x'_\ell) x'^a_i + v$ , where  $v \in \sum_{b < a} \sum_{h, h_1=1}^{\max\{i, \ell\}} \mathbf{f}(x'_h) x'^b_{h_1} R\mathfrak{S}_r$ .*
- (4)  *$\bar{x}'_j \mathbf{f}(x'_i) = \mathbf{f}(x'_i) \bar{x}'^a_j + v$ , where  $v \in \sum_{b_1+b_2 < a, c_1+c_2 \leq 1} \epsilon \bar{x}'^{b_1}_j e^{c_1}_{ij} \mathbf{f}(x'_i) e^{c_2}_{ij} \bar{x}'^{b_2}_j$  for some non-negative integers  $b_1, b_2, c_1, c_2$  and  $\epsilon = \pm 1$ .*

*Proof.* (1)-(2) are trivial. Since  $x_2 = x'_2 - s_1$  and  $x_2 x_1 = x_1 x_2$ ,

$$x'_2 \mathbf{f}(x_1) = \mathbf{f}(x_1)(x'_2 - s_1) + \mathbf{f}(x'_2) s_1. \quad (2.15)$$

Applying the conjugate of  $s_{i,2}$  on (2.15) yields (3) for  $a = 1$  and  $\ell = 1$ . If  $\ell > 1$ , then  $x'_i \mathbf{f}(x'_\ell) = x'_i s_{\ell-1} \mathbf{f}(x'_{\ell-1}) s_{\ell-1} = s_{\ell-1} x'_{(i)s_{\ell-1}} \mathbf{f}(x'_{\ell-1}) s_{\ell-1}$ . Thus, (3) follows from inductive assumption on  $\ell - 1$  and (1) under the assumption  $a = 1$ . The case  $a > 1$  follows by using the previous result on  $a = 1$ , repeatedly. Finally, (4) can be checked similarly by induction. We leave the details to the readers.  $\square$

**Proposition 2.9.** *Let  $J_L = \sum_{i=1}^t \mathcal{B}_{r,t}^{\text{aff}} \mathbf{g}(\bar{x}'_i)$  and  $J_R = \sum_{i=1}^r \mathbf{f}(x'_i) \mathcal{B}_{r,t}^{\text{aff}}$ . Then*

- (1)  *$J_L$  is a right  $R\mathfrak{S}_r \otimes \mathcal{H}_t^{\text{aff}}$ -module;*
- (2)  *$J_R$  is a left  $\mathcal{H}_r^{\text{aff}} \otimes R\tilde{\mathfrak{S}}_t$ -module;*
- (3) *if  $\mathcal{B}_{r,t}^{\text{aff}}$  is admissible, then  $I = J_L + J_R$ , where  $I$  is the two-sided ideal of  $\mathcal{B}_{r,t}^{\text{aff}}$  generated by  $\mathbf{f}(x_1)$  and  $\mathbf{g}(\bar{x}_1)$  satisfying (2.5)-(2.7).*

*Proof.* Obviously, both  $J_L$  and  $J_R$  are  $\mathfrak{S}_r \times \tilde{\mathfrak{S}}_t$ -bimodules. By Lemma 2.8 (3),  $x_1 J_R \subseteq J_R$ . Similarly,  $J_L \bar{x}_1 \subseteq J_L$ . This proves (1)-(2). In order to prove (3), it suffices to verify that  $J_L + J_R$  is a two-sided ideal of  $\mathcal{B}_{r,t}^{\text{aff}}$ . If so, since  $\{\mathbf{f}(x_1), \mathbf{g}(\bar{x}_1)\} \subset J_L + J_R$ ,  $I = J_L + J_R$ , proving the result.

We claim that  $e_1(J_L + J_R) \subseteq J_L + J_R$  and  $(J_L + J_R)e_1 \subseteq J_L + J_R$ . If so, by (2.3),  $(\bar{x}_1 + e_1)\mathbf{f}(x_1) = \mathbf{f}(x_1)(\bar{x}_1 + e_1)$  and hence  $\bar{x}_1 \mathbf{f}(x_1) \in J_L + J_R$ . By (1)-(2),  $\bar{x}_1 \mathbf{f}(x'_i) = s_{i,1} \bar{x}_1 \mathbf{f}(x_1) s_{1,i} \in J_L + J_R$ , and hence  $\bar{x}_1(J_L + J_R) \subseteq J_L + J_R$ . Similarly,  $(J_L + J_R)x_1 \subseteq J_L + J_R$ . Thus the claim implies that  $J_L + J_R$  is a two-sided ideal of  $\mathcal{B}_{r,t}^{\text{aff}}$ .

By symmetry, it remains to prove  $e_1(J_L + J_R) \subseteq J_L + J_R$ . Obviously, it suffices to verify

$$e_1 J_R \subset J_L + J_R. \quad (2.16)$$

By (2.2),  $e_1 \mathbf{f}(x'_i) = \mathbf{f}(x'_i) e_1$  for  $i \geq 2$ . Let  $\mathbf{m}$  be a regular monomial of  $\mathcal{B}_{r,t}^{\text{aff}}$  defined in (2.12). Then  $\mathbf{m} = x^\alpha e_{\vec{i}, \vec{j}} w \bar{x}^\beta$  for some  $w \in \mathfrak{S}_r \times \bar{\mathfrak{S}}_t$ ,  $(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t$  and some  $\vec{i}, \vec{j}$ . Using induction on  $|\alpha|$ , we want to prove

$$e_1 \mathbf{f}(x_1) \mathbf{m} \in J_L + J_R. \quad (2.17)$$

If so, then  $e_1 \mathbf{f}(x_1) \mathcal{B}_{r,t}^{\text{aff}} \subset J_L + J_R$  and hence (2.16) follows.

*Case 1:*  $|\alpha| = 0$ .

If  $f = 0$ , then (2.17) follows from (1) and (2.6). Suppose  $1 \leq f \leq \min\{r, t\}$ . Since  $\mathcal{B}_{r,t}^{\text{aff}}$  is admissible,  $e_1 \mathbf{f}(x_1) \mathbf{m} = 0$  if  $e_i$  is a factor of  $e_{\vec{i}, \vec{j}}$ . Assume that  $e_1$  is not a factor of  $e_{\vec{i}, \vec{j}}$ . If there is an  $l$  such that  $i_l = p \neq 1$  and  $j_l = 1$ , by (2),

$$e_1 \mathbf{f}(x_1) e_{p,1} = s_{p,2} e_1 \mathbf{f}(x_1) s_1 e_{1,p} = s_{p,2} e_1 s_1 \mathbf{f}(x'_2) e_{1,p} = s_{p,2} \mathbf{f}(x'_2) e_{1,p} \in J_R.$$

Suppose  $j_l \neq 1$  for all possible  $l$ . If there is an  $l$  such that  $e_{i_l, j_l} = e_{1,p}$  for some  $p \neq 1$ , then we assume  $i_1 = 1$  and  $j_1 = p$  without loss of any generality. In this case,

$$e_1 \mathbf{f}(x_1) e_{1,p} = (-1)^k \bar{s}_{p,2} e_1 \mathbf{g}(\bar{x}_1) \bar{s}_1 e_{1,p} = (-1)^k \bar{s}_{p,2} e_1 \mathbf{g}(\bar{x}'_2) \bar{s}_{1,p} = (-1)^k \bar{s}_{p,2} e_1 \bar{s}_{1,p} \mathbf{g}(\bar{x}_1).$$

Since  $j_l \neq 1$  for  $1 \leq l \leq f$ , by [19, Lemma 4.7(2)],  $\bar{x}_1 e_{i_l, j_l} = e_{i_l, j_l} \bar{x}_1$  and hence

$$\mathbf{g}(\bar{x}_1) \prod_{l=2}^f e_{i_l, j_l} = \prod_{l=2}^f e_{i_l, j_l} \mathbf{g}(\bar{x}_1) \in J_L. \quad (2.18)$$

Now, (2.17) follows from (1). Finally, if  $\{i_l, j_l\} \cap \{1\} = \emptyset$  for all possible  $l$ , then (2.17) follows from (1) and the following fact

$$e_1 \mathbf{f}(x_1) \prod_{l=1}^f e_{i_f, j_f} = \prod_{l=1}^f e_{i_f, j_f} e_1 \mathbf{f}(x_1) = (-1)^k \prod_{l=1}^f e_{i_f, j_f} e_1 \mathbf{g}(\bar{x}_1) \in J_L.$$

*Case 2:*  $|\alpha| > 0$ .

If  $\alpha_i \neq 0$  for some  $2 \leq i \leq r$ , then  $e_1 x_i = x'_i e_1 - e_1 \sum_{j=1}^i (j, i)$  and  $x_i \mathbf{f}(x_1) = \mathbf{f}(x_1) x_i$ . Let  $\mathbf{m}'$  be obtained from  $\mathbf{m}$  by removing  $x_i$ . Then

$$e_1(1, i) \mathbf{f}(x_1) \mathbf{m}' = e_1 \mathbf{f}(x'_i)(1, i) \mathbf{m}' = \mathbf{f}(x'_i) e_1(1, i) \mathbf{m}' \in J_R.$$

Now, (2.17) follows from inductive assumption on  $|\alpha|$ . If  $\alpha_i = 0$ ,  $2 \leq i \leq r$ , then  $x^\alpha = x_1^{\alpha_1}$  with  $\alpha_1 > 0$ . Let  $v = e_1 \mathbf{f}(x_1) \mathbf{m}$ . If  $j_\ell \neq 1$ ,  $1 \leq \ell \leq f$ , then by (2.18), Lemma 2.8 and inductive assumption,

$$\begin{aligned} v &= e_1 \mathbf{f}(x_1) x_1^{\alpha_1} e_{\vec{i}, \vec{j}} w \bar{x}^\beta = (-1)^k e_1 \mathbf{g}(\bar{x}_1) x_1^{\alpha_1} e_{\vec{i}, \vec{j}} w \bar{x}^\beta \equiv (-1)^k e_1 x_1^{\alpha_1} \mathbf{g}(\bar{x}_1) e_{\vec{i}, \vec{j}} w \bar{x}^\beta \\ &= (-1)^k e_1 x_1^{\alpha_1} e_{\vec{i}, \vec{j}} \mathbf{g}(\bar{x}_1) w \bar{x}^\beta \in J_L w \bar{x}^\beta \subset J_L + J_R, \end{aligned}$$

where the “ $\equiv$ ” is modulo  $J_L + J_R$ . Finally, if  $j_\ell = 1$  for some  $\ell$ , without loss of any generality, we assume  $j_1 = 1$ . If  $i_1 = 1$ , by Lemma 2.4,  $v = e_1 \mathbf{f}(x_1) x_1^{\alpha_1} e_1 e_{\vec{i}, \vec{j}} w \bar{x}^\beta = 0$ , where

$\vec{i}' = (i_2, \dots, i_f)$  and  $\vec{j}' = (j_2, \dots, j_f)$ . Now, we assume  $i_1 \neq 1$ . Then

$$\begin{aligned} v &= e_1 \mathbf{f}(x_1) x_1^{\alpha_1} e_{i_1, 1} e_{\vec{i}', \vec{j}'} w \bar{x}^\beta = e_1 e_{i_1, 1} \mathbf{f}(x_1) x_1^{\alpha_1} e_{\vec{i}', \vec{j}'} w \bar{x}^\beta \\ &= e_1 (1, i_1) \mathbf{f}(x_1) x_1^{\alpha_1} e_{\vec{i}', \vec{j}'} w \bar{x}^\beta = e_1 \mathbf{f}(x'_i) (1, i) x_1^{\alpha_1} e_{\vec{i}', \vec{j}'} w \bar{x}^\beta, \\ &= \mathbf{f}(x'_i) e_1 (1, i) x_1^{\alpha_1} e_{\vec{i}', \vec{j}'} w \bar{x}^\beta \in J_R. \end{aligned}$$

This completes the proof of (2.17).  $\square$

For  $(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t$ , denote  $\mathbf{f}(x')^\alpha = \mathbf{f}(x_1)^{\alpha_1} \cdots \mathbf{f}(x'_r)^{\alpha_r}$  and  $\mathbf{g}(\bar{x}')^\beta = \mathbf{g}(\bar{x}_1)^{\beta_1} \cdots \mathbf{g}(\bar{x}'_t)^{\beta_t}$ . Let  $\mathbb{N}_k^r = \{\alpha \in \mathbb{N}^r \mid \alpha_i \leq k-1, 1 \leq i \leq r\}$  and  $\mathbb{N}_k^t = \{\alpha \in \mathbb{N}^t \mid \alpha_i \leq k-1, 1 \leq i \leq t\}$ .

**Lemma 2.10.** *The affine walled Brauer algebra  $\mathcal{B}_{r,t}^{\text{aff}}$  is a free  $R$ -module with  $\mathcal{N}$  as its  $R$ -basis, where*

$$\mathcal{N} = \bigcup_{f=0}^{\min\{m,n\}} \{ \mathbf{f}(x')^\alpha x^\gamma c^{-1} e^f w d \bar{x}^\delta \mathbf{g}(\bar{x}')^\beta \mid (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t, (\gamma, \delta) \in \mathbb{N}_k^r \times \mathbb{N}_k^t, c, d \in \mathcal{D}_{r,t}^f, w \in \mathfrak{S}_{r-f} \times \tilde{\mathfrak{S}}_{t-f} \}. \quad (2.19)$$

*Proof.* The result follows from Theorem 2.6 since the transition matrix between  $\mathcal{N}$  and  $\mathcal{M}$  in (2.12) is invertible.  $\square$

**Lemma 2.11.** *Let  $I$  be the two-sided ideal of  $\mathcal{B}_{r,t}^{\text{aff}}$  generated by  $\mathbf{f}(x_1)$  and  $\mathbf{g}(\bar{x}_1)$  satisfying (2.5)–(2.7). If  $\mathcal{B}_{r,t}^{\text{aff}}$  is admissible, then  $S$  is an  $R$ -basis of  $I$ , where*

$$S = \{ \mathbf{f}(x')^\alpha x^\gamma c^{-1} e^f w d \bar{x}^\delta \mathbf{g}(\bar{x}')^\beta \in \mathcal{N} \mid \alpha_i + \beta_j \neq 0 \text{ for some } i, j \}. \quad (2.20)$$

*Proof.* Let  $M = \text{span}_R S$ . By Lemma 2.10,  $\mathbf{f}(x_1) \mathcal{B}_{r,t}^{\text{aff}} \subseteq M$ . For any positive integer  $l$  with  $1 \leq l < i$ , by Lemma 2.8 (2),

$$\mathbf{f}(x'_i) \mathbf{f}(x'_l) \in \sum_{j=1}^{i-1} \mathbf{f}(x'_j) \mathcal{B}_{r,t}^{\text{aff}} + \mathbf{f}(x'_i) D,$$

such that  $D \in \mathcal{B}_{r,t}^{\text{aff}}$  and the degree of  $D$  is strictly less than  $k$ . Thus,  $\mathbf{f}(x'_i) \mathcal{B}_{r,t}^{\text{aff}} \subseteq M$  which follows from inductive assumption on  $j$  with  $1 \leq j \leq i-1$  and inductive assumption on degrees. This proves  $J_R \subseteq M$ . One can check  $J_L \subseteq M$  similarly. By Proposition 2.9 (3),  $I = M$ .  $\square$

By abuse of notions, a regular monomial  $\mathbf{m}$  in Definition 2.5 is also called a *regular monomial* of  $\mathcal{B}_{k,r,t}$  if  $0 \leq \alpha_i, \beta_j \leq k-1$  for all  $i, j$  with  $1 \leq i \leq r$  and  $1 \leq j \leq t$ . Obviously, the number of all such regular monomials is  $k^{r+t}(r+t)!$ .

**Theorem 2.12.** *The cyclotomic walled Brauer algebra  $\mathcal{B}_{k,r,t}$  is free over  $R$  with rank  $k^{r+t}(r+t)!$  if and only if  $\mathcal{B}_{k,r,t}$  is admissible.*

*Proof.* Let  $M$  be the  $R$ -submodule of  $\mathcal{B}_{k,r,t}$  spanned by all regular monomials of  $\mathcal{B}_{k,r,t}$ . By induction on degrees, it is routine to check that  $M$  is left  $\mathcal{B}_{k,r,t}$ -module (cf. [19, Proposition 4.12] for  $\mathcal{B}_{r,t}^{\text{aff}}$ ). Since  $1 \in M$ , we have  $M = \mathcal{B}_{k,r,t}$ . If  $\mathcal{B}_{k,r,t}$  is not admissible, by Lemma 2.2,  $e_1$  is an  $R$ -torsion element. Since  $e_1 \in M$ , either  $\mathcal{B}_{k,r,t}$  is not free or the rank of  $\mathcal{B}_{k,r,t}$  is strictly less than  $k^{r+t}(r+t)!$ . If  $\mathcal{B}_{k,r,t}$  is admissible, by Lemmas 2.10–2.11, the set of all regular monomials of  $\mathcal{B}_{k,r,t}$  is  $R$ -linear independent. Thus,  $\mathcal{B}_{k,r,t}$  is free over  $R$  with rank  $k^{r+t}(r+t)!$ .  $\square$



3. A WEAKLY CELLULAR BASIS OF  $\mathcal{B}_{2,r,t}$ 

The aim of this section is to construct a weakly cellular basis of  $\mathcal{B}_{2,r,t}$  in the sense of [11]. This basis will be used to set up a relationship between  $\mathfrak{gl}_{m|n}$ -Kac-modules and right cell modules of  $\mathcal{B}_{2,r,t}$  in section 6.

Recall that a *composition* of  $r$  is a sequence of non-negative integers  $\tau = (\tau_1, \tau_2, \dots)$  such that  $|\tau| := \sum_i \tau_i = r$ . If  $\tau_i \geq \tau_{i+1}$  for all possible  $i$ 's, then  $\tau$  is called a *partition*. Similarly, a  $k$ -*partition* of  $r$ , or simply a *multipartition* of  $r$ , is an ordered  $k$ -tuple  $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$  of partitions with  $|\lambda| := \sum_{i=1}^k |\lambda^{(i)}| = r$ . Let  $\Lambda_k^+(r)$  be the set of all  $k$ -partitions of  $r$ . Let  $\leq$  be the dominant order defined on  $\Lambda_k^+(n)$  in the sense that  $\lambda \leq \mu$  if and only if

$$\sum_{h=1}^{\ell-1} |\lambda^{(h)}| + \sum_{j=1}^i \lambda_j^{(\ell)} \leq \sum_{h=1}^{\ell-1} |\mu^{(h)}| + \sum_{j=1}^i \mu_j^{(\ell)} \text{ for } \ell \leq k \text{ and all possible } i, \quad (3.1)$$

where  $|\lambda^{(0)}| = 0$ . Then  $\Lambda_k^+(r)$  is a poset with  $\leq$  as a partial order on it. In this paper, we always assume  $k \in \{1, 2\}$ .

For each  $\lambda \in \Lambda_1^+(r)$ , the *Young diagram*  $[\lambda]$  is a collection of boxes arranged in left-justified rows with  $\lambda_i$  boxes in the  $i$ -th row of  $[\lambda]$ . A  $\lambda$ -*tableau*  $\mathfrak{s}$  is obtained by inserting elements  $i$ ,  $1 \leq i \leq r$  into  $[\lambda]$  without repetition. A  $\lambda$ -tableau  $\mathfrak{s}$  is said to be *standard* if the entries in  $\mathfrak{s}$  increase both from left to right in each row and from top to bottom in each column. Let  $\mathcal{T}^s(\lambda)$  be the set of all standard  $\lambda$ -tableaux. Let  $\mathfrak{t}^\lambda \in \mathcal{T}^s(\lambda)$  be obtained from  $[\lambda]$  by adding  $1, 2, \dots, r$  from left to right along the rows of  $[\lambda]$ . Let  $\mathfrak{t}_\lambda \in \mathcal{T}^s(\lambda)$  be obtained from  $[\lambda]$  by adding  $1, 2, \dots, r$  from top to bottom along the columns of  $[\lambda]$ . For example, if  $\lambda = (3, 2)$ , then

$$\mathfrak{t}^\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}, \quad \text{and} \quad \mathfrak{t}_\lambda = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}. \quad (3.2)$$

If  $\lambda \in \Lambda_2^+(r)$ , then the corresponding Young diagram  $[\lambda]$  is  $([\lambda^{(1)}], [\lambda^{(2)}])$ . In this case, a  $\lambda$ -*tableau*  $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$  is obtained by inserting elements  $i$ ,  $1 \leq i \leq r$  into  $[\lambda]$  without repetition. A  $\lambda$ -tableau  $\mathfrak{s}$  is said to be *standard* if the entries in  $\mathfrak{s}_i$ ,  $1 \leq i \leq 2$  increase both from left to right in each row and from top to bottom in each column. Let  $\mathcal{T}^s(\lambda)$  be the set of all standard  $\lambda$ -tableaux. Let  $\mathfrak{t}^\lambda \in \mathcal{T}^s(\lambda)$  be obtained from  $[\lambda]$  by adding  $1, 2, \dots, r$  from left to right along the rows of  $[\lambda^{(1)}]$  and then  $[\lambda^{(2)}]$ . Let  $\mathfrak{t}_\lambda \in \mathcal{T}^s(\lambda)$  be obtained from  $[\lambda]$  by adding  $1, 2, \dots, r$  from top to bottom along the columns of  $[\lambda^{(2)}]$  and then  $[\lambda^{(1)}]$ . For example, if  $\lambda = ((3, 2), (3, 1)) \in \Lambda_2^+(9)$ , then

$$\mathfrak{t}^\lambda = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 6 & 7 & 8 \\ \hline 9 & & \\ \hline \end{array} \right) \quad \text{and} \quad \mathfrak{t}_\lambda = \left( \begin{array}{|c|c|c|} \hline 5 & 7 & 9 \\ \hline 6 & 8 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array} \right). \quad (3.3)$$

Recall that  $\mathfrak{S}_r$  acts on the right of  $1, 2, \dots, r$ . Then  $\mathfrak{S}_r$  acts on the right of a  $\lambda$ -tableau  $\mathfrak{s}$  by permuting its entries. For example, if  $\lambda = ((3, 2), (3, 1)) \in \Lambda_2^+(9)$ , and  $w = s_1 s_2$ , then

$$\mathfrak{t}^\lambda w = \left( \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline 4 & 5 & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 6 & 7 & 8 \\ \hline 9 & & \\ \hline \end{array} \right). \quad (3.4)$$

Write  $d(\mathfrak{s}) = w$  for  $w \in \mathfrak{S}_r$  if  $\mathfrak{t}^\lambda w = \mathfrak{s}$ . Then  $d(\mathfrak{s})$  is uniquely determined by  $\mathfrak{s}$ . Let  $w_\lambda = d(\mathfrak{t}_\lambda)$ . The row stabilizer  $\mathfrak{S}_\lambda$  of  $\mathfrak{t}^\lambda$  for  $\lambda \in \Lambda_k^+(r)$  is known as the Young subgroup of

$\mathfrak{S}_r$  with respect to  $\lambda$ . It is the same as the Young subgroup  $\mathfrak{S}_{\lambda_{\text{comp}}}$  with respect to the composition  $\lambda_{\text{comp}}$ , which is obtained from  $\lambda$  by concatenation. For example,  $\lambda_{\text{comp}} = (3, 2, 3, 1)$  if  $\lambda = ((3, 2), (3, 1))$ .

The level two degenerate Hecke  $\mathcal{H}_{2,r}$  with defining parameters  $u_1$  and  $u_2$  is  $\mathcal{H}_r^{\text{aff}}/I$ , where  $I$  is the two-sided ideal of  $\mathcal{H}_r^{\text{aff}}$  generated by  $(y_1 - u_1)(y_1 - u_2)$ ,  $u_1, u_2 \in R$ . By definition,  $\mathcal{H}_{2,r}$  is an  $R$ -algebra generated by  $s_i$ ,  $1 \leq i \leq r-1$  and  $y_j$ ,  $1 \leq j \leq r$  such that

- (1)  $s_i s_j = s_j s_i$ ,  $1 < |i - j|$ ,
- (2)  $y_i y_\ell = y_\ell y_i$ ,  $1 \leq i, \ell \leq r$ ,
- (3)  $s_i y_i - y_{i+1} s_i = -1$ ,  $y_i s_i - s_i y_{i+1} = -1$ ,  $1 \leq i \leq r-1$ ,
- (4)  $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ ,  $1 \leq j \leq r-2$ ,
- (5)  $s_i^2 = 1$ ,  $1 \leq i \leq r-1$ ,
- (6)  $(y_1 - u_1)(y_1 - u_2) = 0$ .

Following [3], we define  $\pi_\lambda = \pi_a(u_2)$  and  $\tilde{\pi}_\lambda = \pi_a(u_1)$  for  $\lambda \in \Lambda_2^+(r)$  with  $|\lambda^{(1)}| = a$ , where for any  $u \in R$ ,  $\pi_0(u) = 1$  and  $\pi_a(u) = \prod_{i=1}^a (y_i - u)$  if  $a > 0$ . Let

$$w_a = \begin{pmatrix} 1 & 2 & \cdots & a & a+1 & a+2 & \cdots & r \\ r-a+1 & r-a+3 & \cdots & r & 1 & 2 & \cdots & r-a \end{pmatrix}. \quad (3.5)$$

It is well-known that

$$w_a s_j = s_{(j)w_a^{-1}} w_a, \quad \text{if } j \neq r-a. \quad (3.6)$$

Let  $\mathfrak{S}_{a,r-a}$  be the Young subgroup with respect to the composition  $(a, r-a)$ . Then

$$R\mathfrak{S}_{a,r-a} w_a = w_a R\mathfrak{S}_{r-a,a}. \quad (3.7)$$

For each composition  $\lambda$  of  $r$ , we denote

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w, \quad y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-1)^{\ell(w)} w, \quad (3.8)$$

where  $\ell(\cdot)$  is the length function on  $\mathfrak{S}_r$ . Assume  $\lambda \in \Lambda_2^+(r)$  with  $|\lambda^{(1)}| = a$ . If we denote  $\mu^{(i)} = (\lambda^{(i)})'$ , the conjugate of  $\lambda^{(i)}$  for  $i = 1, 2$ , then

$$w_a x_{\mu^{(2)}} y_{\mu^{(1)}} = y_{\mu^{(1)}} x_{\mu^{(2)}} w_a. \quad (3.9)$$

*Remark 3.1.* When we write  $x_{\mu^{(2)}} y_{\mu^{(1)}}$ , then  $x_{\mu^{(2)}}$  (resp.,  $y_{\mu^{(1)}}$ ) is defined via symmetric group on  $r-a$  letters  $\{1, 2, \dots, r-a\}$  (resp., on  $a$  letters  $\{r-a+1, \dots, r\}$ ). Similarly, when we write  $y_{\mu^{(1)}} x_{\mu^{(2)}}$ , then  $y_{\mu^{(1)}}$  (resp.,  $x_{\mu^{(2)}}$ ) is defined via symmetric group on  $a$  letters  $\{1, 2, \dots, a\}$  (resp., on  $r-a$  letters  $\{a+1, a+2, \dots, r\}$ ).

**Definition 3.2.** For any  $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)$  with  $\lambda \in \Lambda_2^+(r)$ , define

- (1)  $\mathfrak{x}_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s})^{-1} \mathfrak{x}_\lambda d(\mathfrak{t})$ , where  $\mathfrak{x}_\lambda = \pi_\lambda x_{\lambda^{(1)}} y_{\lambda^{(2)}}$ ,
- (2)  $\mathfrak{y}_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s})^{-1} \mathfrak{y}_\lambda d(\mathfrak{t})$ , where  $\mathfrak{y}_\lambda = \tilde{\pi}_\lambda x_{\lambda^{(1)}} y_{\lambda^{(2)}}$ ,
- (3)  $\bar{\mathfrak{x}}_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s})^{-1} \bar{\mathfrak{x}}_\lambda d(\mathfrak{t})$ , where  $\bar{\mathfrak{x}}_\lambda = \pi_\lambda y_{\lambda^{(1)}} x_{\lambda^{(2)}}$ ,
- (4)  $\bar{\mathfrak{y}}_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s})^{-1} \bar{\mathfrak{y}}_\lambda d(\mathfrak{t})$ , where  $\bar{\mathfrak{y}}_\lambda = \tilde{\pi}_\lambda y_{\lambda^{(1)}} x_{\lambda^{(2)}}$ .

It is proven in [3] that  $\mathcal{H}_{2,r}$  is a cellular algebra over  $R$  in the sense of [12]. In this paper, we need the following cellular basis of  $\mathcal{H}_{2,r}$  so as to construct a new weakly cellular basis of  $\mathcal{B}_{2,r,t}$ .

**Lemma 3.3.** *The set  $S_i$ ,  $i \in \{1, 2, 3, 4\}$ , are cellular bases of  $\mathcal{H}_{2,r}$  in the sense of [12], where*

- (1)  $S_1 = \{\mathfrak{x}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\},$
- (2)  $S_2 = \{\mathfrak{y}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\},$
- (3)  $S_3 = \{\bar{\mathfrak{x}}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\},$
- (4)  $S_4 = \{\bar{\mathfrak{y}}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda_2^+(r), \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda)\}.$

*Proof.* Let  $S = \{x_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\lambda), \lambda \in \Lambda_2^+(r)\}$  and  $x_{\mathfrak{s}\mathfrak{t}} = d(\mathfrak{s})^{-1} \pi_\lambda x_{\lambda^{(1)}} x_{\lambda^{(2)}} d(\mathfrak{t})$ . It is proven in [3] that  $S$  is a cellular basis of  $\mathcal{H}_{2,r}$ . If we use  $y_{\lambda^{(2)}}$  instead of  $x_{\lambda^{(2)}}$  in  $x_{\mathfrak{s}\mathfrak{t}}$ , we will get  $\mathfrak{x}_{\mathfrak{s}\mathfrak{t}}$ . However, for any  $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{T}^s(\lambda)$ ,  $d(\mathfrak{s})$  can be written uniquely as  $d(\mathfrak{s}_1)d(\mathfrak{s}_2)d$  such that  $d$  is a distinguished right coset representative of  $\mathfrak{S}_a \times \mathfrak{S}_{r-a}$  in  $\mathfrak{S}_r$  and  $\mathfrak{s}_i \in \mathcal{T}^s(\lambda^{(i)})$ , where  $a = |\lambda^{(1)}|$ . So, the transition matrix between  $S_1$  and  $S$  is determined by the transition matrix between the cellular basis  $\{d(\mathfrak{s}_2)^{-1} x_{\lambda^{(2)}} d(\mathfrak{t}_2) \mid \lambda^{(2)} \in \Lambda^+(r-a), \mathfrak{s}_2, \mathfrak{t}_2 \in \mathcal{T}^s(\lambda^{(2)})\}$  and  $\{d(\mathfrak{s}_2)^{-1} y_{\lambda^{(2)}} d(\mathfrak{t}_2) \mid \lambda^{(2)} \in \Lambda^+(r-a), \mathfrak{s}_2, \mathfrak{t}_2 \in \mathcal{T}^s(\lambda^{(2)})\}$  of  $R\mathfrak{S}_{r-a}$ . Thus,  $S_1$  is a basis of  $\mathcal{H}_{2,r}$ . One can check that  $S_1$  is a cellular basis of  $\mathcal{H}_{2,r}$  in the sense of [12] by mimicking Dipper-James-Murphy's arguments in the proof of Murphy basis for Hecke algebras of type  $B$  in [8]. We leave the details to the readers. Finally, (2)–(4) can be verified similarly.  $\square$

By Graham-Lehrer's results on the representation theory of cellular algebras in [12], one can define right cell modules of  $\mathcal{H}_{2,r}$  via the cellular bases  $S_i$ ,  $i \in \{1, 2, 3, 4\}$  in Lemma 3.3. The corresponding right cell modules of  $\mathcal{H}_{2,r}$  with respect to  $S_2$  and  $S_4$  are denoted by  $\tilde{\Delta}(\lambda)$ , and  $\bar{\Delta}(\lambda)$ .

For the simplification of discussion, we assume  $\mathcal{H}_{2,r}$  is defined over  $\mathbb{C}$  in Lemma 3.4.

**Lemma 3.4.** *Suppose  $a, b \in \mathbb{N}$ . Then*

- (1)  $\pi_a(u_2)\mathcal{H}_{2,r}\pi_b(u_1) = 0$  whenever  $a + b > r$  and  $a, b \in \mathbb{Z}^{>0}$ .
- (2)  $\pi_a(u_2)\mathcal{H}_{2,r}\pi_{r-a}(u_1) = \pi_a(u_2)w_a\pi_{r-a}(u_1)\mathbb{C}\mathfrak{S}_{r-a,a}$ , where  $\mathfrak{S}_{r-a,a}$  is as in (3.7).
- (3)  $\mathfrak{x}_\lambda\mathcal{H}_{2,r}\mathfrak{y}_{\mu'} = 0$  if  $\lambda, \mu \in \Lambda_2^+(r)$  with  $\lambda \triangleright \mu$ ,
- (4)  $\mathfrak{x}_\lambda\mathcal{H}_{2,r}\mathfrak{y}_{\lambda'} = \text{Span}_{\mathbb{C}}\{\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\}$  if  $\lambda \in \Lambda_2^+(r)$ .
- (5)  $\tilde{\Delta}(\lambda') \cong \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathcal{H}_{2,r}$ .

*Proof.* (1)–(4) can be proven by arguments similar to those for Hecke algebras of type  $B$  in [7]. We only give details for (3) and (5).

If  $\lambda \triangleright \mu$ , then  $|\lambda^{(1)}| \geq |\mu^{(1)}|$ . If  $|\lambda^{(1)}| > |\mu^{(1)}|$ , then  $|\mu^{(1)}| \neq r$  and the result follows from (1). When  $|\lambda^{(1)}| = |\mu^{(1)}|$ , by (2) together with corresponding result for the group algebras of symmetric groups, we have  $\lambda^{(i)} \leq \mu^{(i)}$  for  $i = 1, 2$  if  $\mathfrak{x}_\lambda\mathcal{H}_{2,r}\mathfrak{y}_{\mu'} \neq 0$ . This proves (3).

There is a surjective  $\mathcal{H}_{2,r}$ -homomorphism from  $\phi : \mathfrak{y}_{\lambda'}\mathcal{H}_{2,r} \rightarrow \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathcal{H}_{2,r}$ . Let  $\mathcal{H}_{2,r}^{\triangleright \lambda'}$  be the  $\mathbb{C}$ -submodule spanned by  $\{\mathfrak{y}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^s(\mu), \mu \triangleright \lambda'\}$ . It follows from standard results on cellular algebras that  $\mathcal{H}_{2,r}^{\triangleright \lambda'}$  is a two-sided ideal of  $\mathcal{H}_{2,r}$ . So,  $\mathfrak{y}_{\lambda'}\mathcal{H}_{2,r} + \mathcal{H}_{2,r}^{\triangleright \lambda'} / \mathcal{H}_{2,r}^{\triangleright \lambda'}$  is isomorphic to a submodule of  $\tilde{\Delta}(\lambda')$ . If  $\mathfrak{y}_{\mathfrak{s}\mathfrak{t}} \in \mathcal{H}_{2,r}^{\triangleright \lambda'}$ , we have  $\mu \triangleright \lambda'$  which is equivalent to  $\lambda \triangleright \mu'$ . By (3),  $x_\lambda w_\lambda \mathfrak{y}_{\mathfrak{s}\mathfrak{t}} = 0$  and  $\mathcal{H}_{2,r}^{\triangleright \lambda'} \subset \ker \phi$ . So, there is an epimorphism from  $\mathfrak{y}_{\lambda'}\mathcal{H}_{2,r} + \mathcal{H}_{2,r}^{\triangleright \lambda'} / \mathcal{H}_{2,r}^{\triangleright \lambda'}$  to  $\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathcal{H}_{2,r}$ . Mimicking arguments on classical Specht modules for Hecke algebra of type  $B$  in [7], we know that  $\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathcal{H}_{2,r}$  has a basis  $\{\mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'} d(\mathfrak{t}) \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$ . So,

$$\dim_{\mathbb{C}} \tilde{\Delta}(\lambda') = \dim_{\mathbb{C}} \mathfrak{x}_\lambda w_\lambda \mathfrak{y}_{\lambda'}\mathcal{H}_{2,r} = \#\mathcal{T}^s(\lambda'),$$

forcing  $\eta_{\lambda'} \mathcal{H}_{2,r} + \mathcal{H}_{2,r}^{\triangleright \lambda'} / \mathcal{H}_{2,r}^{\triangleright \lambda'} \cong \mathfrak{r}_{\lambda} w_{\lambda} \eta_{\lambda'} \mathcal{H}_{2,r} \cong \tilde{\Delta}(\lambda')$ .  $\square$

Now, we use cellular bases  $S_i$  of  $\mathcal{H}_{2,r}$  in Lemma 3.3 to construct a weakly cellular basis of  $\mathcal{B}_{2,r,t}$  over an arbitrary field in the sense of [11]. We remark that when we use results on level two degenerate Hecke algebra for  $\mathcal{B}_{2,r,t}$ , we should keep in mind that  $x_1, \bar{x}_1 \in \mathcal{B}_{2,r,t}$  should be regarded as  $-y_1 \in \mathcal{H}_{2,r}$  and  $\mathcal{H}_{2,t}$ , respectively. Therefore, we have to use  $-u_i$  and  $-\bar{u}_i$  instead of  $u_i$  and  $\bar{u}_i$ .

Fix  $r, t, f \in \mathbb{Z}^{>0}$  with  $f \leq \min\{r, t\}$ . In contrast to (2.11), we define

$$\mathcal{D}_{r,t}^f = \{s_{r-f+1, i_{r-f+1}} \bar{s}_{t-f+1, j_{t-f+1}} \cdots s_{r, i_r} \bar{s}_{t, j_t} \mid r \geq i_r > \cdots > i_{r-f+1}, j_k \geq k + f - t\}. \quad (3.10)$$

For each  $c \in \mathcal{D}_{r,t}^f$  as in (3.10), let  $\kappa_c$  be the  $r$ -tuple

$$\kappa_c = (k_1, \dots, k_r) \in \{0, 1\}^r \text{ such that } k_i = 0 \text{ unless } i = i_r, i_{r-1}, \dots, i_{r-f+1}. \quad (3.11)$$

Note that  $\kappa_c$  may have more than one choice for a fixed  $c$ , and it may be equal to  $\kappa_d$  although  $c \neq d$  for  $c, d \in \mathcal{D}_{r,t}^f$ . Let  $\mathbf{N}_f = \{\kappa_c \mid c \in \mathcal{D}_{r,t}^f\}$ . If  $\kappa_c \in \mathbf{N}_f$ , define  $x^{\kappa_c} = \prod_{i=1}^r x_i^{k_i}$ . In [19], we consider poset  $(\Lambda_{2,r,t}, \triangleright)$ , where

$$\Lambda_{2,r,t} = \{(f, \lambda, \mu) \mid (\lambda, \mu) \in \Lambda_2^+(r-f) \times \Lambda_2^+(t-f), 0 \leq f \leq \min\{r, t\}\}, \quad (3.12)$$

such that  $(f, \lambda, \mu) \triangleright (\ell, \alpha, \beta)$  for  $(f, \lambda, \mu), (\ell, \alpha, \beta) \in \Lambda_{2,r,t}$  if either  $f > \ell$  or  $f = \ell$  and  $\lambda \triangleright_1 \alpha$ , and  $\mu \triangleright_2 \beta$ , and in case  $f = \ell$ , the orders  $\triangleright_1$  and  $\triangleright_2$  are dominant orders on  $\Lambda_2^+(r-f)$  and  $\Lambda_2^+(t-f)$  respectively. For each  $(f, \mu, \nu) \in \Lambda_{2,r,t}$ , let

$$\delta(f, \mu, \nu) = \{(\mathfrak{t}, c, \kappa_c) \mid \mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}) \in \mathcal{T}^s(\mu) \times \mathcal{T}^s(\nu), c \in \mathcal{D}_{r,t}^f \text{ and } \kappa_c \in \mathbf{N}_f\}. \quad (3.13)$$

**Definition 3.5.** For any  $(\mathfrak{s}, d, \kappa_d), (\mathfrak{t}, c, \kappa_c) \in \delta(f, \mu, \nu)$  with  $(f, \mu, \nu) \in \Lambda_{2,r,t}$ , define

$$C_{(\mathfrak{s}, d, \kappa_d)(\mathfrak{t}, c, \kappa_c)} = x^{\kappa_d} d^{-1} \mathfrak{e}^f \mathbf{n}_{\mathfrak{st}} c x^{\kappa_c}, \quad (3.14)$$

where, in contrast to notation  $e^f$  in (2.10), we define  $\mathfrak{e}^f = e_{r,t} e_{r-1, t-1} \cdots e_{r-f+1, t-f+1}$  if  $f \geq 1$  and  $\mathfrak{e}^0 = 1$ , and  $\mathbf{n}_{\mathfrak{st}} = \eta_{\mathfrak{s}^{(1)} \mathfrak{t}^{(1)}} \bar{\eta}_{\mathfrak{s}^{(2)} \mathfrak{t}^{(2)}}$  if  $\mathfrak{s} = (\mathfrak{s}^{(1)}, \mathfrak{s}^{(2)})$  and  $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)})$  are in  $\mathcal{T}^s(\mu) \times \mathcal{T}^s(\nu)$ .

Note that  $\mathbf{n}_{\mathfrak{st}}$  in Definition 3.5 are defined via cellular basis elements of  $\mathcal{H}_{2, r-f}$  and  $\mathcal{H}_{2, t-f}$  in Lemma 3.3 (2) (4). Since  $x_i$  and  $\bar{x}_j$  do not commute each other, a cellular basis element of  $\mathcal{H}_{2, r-f}$  is always put on the left. Further, we need to use  $x_i, -u_1, -u_2$  (resp.  $\bar{x}_i, -\bar{u}_1, -\bar{u}_2$ ) instead of  $-y_i, u_1, u_2$  in Lemma 3.3.

**Theorem 3.6.** *If  $\mathcal{B}_{2,r,t}$  is admissible, then the set*

$$\mathcal{C} = \{C_{(\mathfrak{s}, \kappa_c, c)(\mathfrak{t}, \kappa_d, d)} \mid (\mathfrak{s}, \kappa_c, c), (\mathfrak{t}, \kappa_d, d) \in \delta(f, \lambda), \forall (f, \lambda) \in \Lambda_{2,r,t}\}$$

*is a weakly cellular basis  $\mathcal{B}_{2,r,t}$  over  $R$  in the sense of [11].*

*Proof.* Let  $S$  be the cellular basis of  $\mathcal{H}_{2, r-f}$  (resp.  $\mathcal{H}_{2, t-f}$ ) for  $0 \leq f \leq \min\{r, t\}$  defined in the proof of Lemma 3.3. If we use  $S$  instead of the cellular basis  $S_2$  of  $\mathcal{H}_{2, r-f}$  and  $S_4$  of  $\mathcal{H}_{2, t-f}$  in Lemma 3.3, we will obtain the weakly cellular basis of  $\mathcal{B}_{2,r,t}$  over  $R$  in [19, Theorem 6.12] provided that  $R = \mathbb{C}$  and  $u_1 = -p, u_2 = m - q, \bar{u}_1 = q$  and  $\bar{u}_2 = p - n$  with  $r + t \leq \min\{m, n\}$ . Since  $\mathcal{B}_{2,r,t}$  is admissible, by Theorem 2.12, the rank of  $\mathcal{B}_{2,r,t}$  is  $2^{r+t}(k+t)!$ . As pointed in [19, Remark 6.13], [19, Theorem 6.12] holds over  $R$  with arbitrary

parameters  $u_1, u_2, \bar{u}_1, \bar{u}_2$  if the rank of  $\mathcal{B}_{2,r,t}$  is  $2^{r+t}(r+t)!$ . Thus,  $\mathcal{C}$  is an  $R$ -basis of  $\mathcal{B}_{2,r,t}$ . Further, the weakly cellularity of  $\mathcal{B}_{2,r,t}$  depends only on cellular bases of  $\mathcal{H}_{2,r-f}$  and  $\mathcal{H}_{2,t-f}$  and does not depend on the explicit descriptions of cellular bases of  $\mathcal{H}_{2,r-f}$  and  $\mathcal{H}_{2,t-f}$ . (cf. the proof of [19, Theorem 6.12]). So, all arguments for the proof of [19, Theorem 6.12] can be used smoothly to prove that  $\mathcal{C}$  is a weakly cellular basis  $\mathcal{B}_{2,r,t}$  over  $R$ .  $\square$

Suppose  $\mathcal{B}_{2,r,t}$  is defined over a field  $F$ . By Theorem 3.6, one can define right cell modules  $C(f, \mu, \nu)$  with respect to  $(f, \mu, \nu) \in \Lambda_{2,r,t}$  for  $\mathcal{B}_{2,r,t}$ . Let  $\phi_{f,\mu,\nu}$  be the corresponding invariant form on  $C(f, \mu, \nu)$  and let  $D^{f,\mu,\nu} = C(f, \mu, \nu)/\text{Rad } \phi_{f,\mu,\nu}$ , where  $\text{Rad } \phi_{f,\mu,\nu}$  is the radical of  $\phi_{f,\mu,\nu}$ . By Graham-Lehrer's results in [12] (a weakly cellular algebra has similar representation theory of a cellular algebra in [12]),  $D^{f,\mu,\nu}$  is either 0 or irreducible and all non-zero  $D^{f,\mu,\nu}$  consist of a complete set of pair-wise non-isomorphic irreducible  $\mathcal{B}_{2,r,t}$ -modules. Let  $\tilde{\Delta}(\mu)$  (resp.  $\bar{\Delta}(\nu)$ ) be the cell module of  $\mathcal{H}_{2,r-f}$  (resp.  $\mathcal{H}_{2,t-f}$ ) defined via  $S_2$  and  $S_4$  in Lemma 3.3. Similarly, one has the notations  $D^\mu$  and  $\bar{D}^\nu$ , respectively.

**Proposition 3.7.** *Suppose that  $\mathcal{B}_{2,r,t}$  is admissible over  $F$ . For any  $(f, \mu, \nu) \in \Lambda_{2,r,t}$ ,  $D^{f,\mu,\nu} \neq 0$  if and only if*

- (1)  $D^\mu \neq 0$  and  $\bar{D}^\nu \neq 0$ ,
- (2)  $f \neq r$  provided  $r = t$  and  $\omega_0 = \omega_1 = 0$ .

*Proof.* The result can be proven by arguments similar to those for Lemmas 7.3–7.4 in [19].  $\square$

*Remark 3.8.* By arguments similar to those for Theorem 3.6, one can lift cellular bases of  $\mathcal{H}_{k,r}$  and  $\mathcal{H}_{k,t}$  in [3] to obtain a weakly cellular basis of  $\mathcal{B}_{k,r,t}$  over  $R$ , provide that  $\mathcal{B}_{k,r,t}$  is admissible. Further, it is not difficult to prove a result which is similar to Proposition 3.7 for  $\mathcal{B}_{k,r,t}$  over an arbitrary field  $F$  with characteristic  $\text{char } F$  either zero or positive. Let  $\mathbf{u} = (u_1, \dots, u_k) \in F^k$  such that  $u_i = d_i \cdot 1_F$  and  $0 \leq d_i < \text{char } F$  for  $1 \leq i \leq k$ . Kleshchev [15] has shown that the simple  $\mathcal{H}_{k,n}(\mathbf{u})$ -modules are labeled by a set of multipartitions which gives the same Kashiwara crystal as the set of  $\mathbf{u}$ -Kleshchev multipartitions of  $n$  in [1, 2]. Thus, the simple  $\mathcal{B}_{k,r,t}$ -modules are labeled by the set  $\{(f, \mu, \nu)\}$ , where (1)  $0 \leq f \leq \min\{r, t\}$ , (2)  $\mu$ 's are Kleshchev multipartitions of  $r-f$  with respect to  $\mathbf{u}$ , (3)  $\nu$ 's are Kleshchev multipartitions of  $t-f$  with respect to  $\bar{\mathbf{u}} := (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_k)$ , (4)  $f \neq r$  if  $r = t$  and  $\omega_i = 0$  for  $0 \leq i \leq k-1$ . It is pointed in [3] that one can modify the proof of [9, Theorem 1.1], or [1, Theorem 1.3], to show that when  $\mathcal{B}_{k,r,t}$  is admissible, the simple  $\mathcal{B}_{k,r,t}$ -modules are always labeled by the  $(f, \mu, \nu) \in \Lambda_{k,r,t}$  with  $0 \leq f \leq \min\{r, t\}$  and  $\mu$  (resp.  $\nu$ ) are Kleshchev multipartitions with respect to  $\mathbf{u}$  (resp.  $\bar{\mathbf{u}}$ ) and  $f \neq r$  if  $r = t$  and  $\omega_i = 0$  for  $1 \leq i \leq r$ . However, we are not claiming that  $D^{(f,\mu,\nu)} \neq 0$  for the multipartitions  $\mu, \nu$  which Kleshchev [15] uses to label the simple  $\mathcal{H}_{k,r-f}(\mathbf{u})$ -modules (resp.  $\mathcal{H}_{k,r-f}(\bar{\mathbf{u}})$ -modules).

We recall the definition of Kleshchev bipartitions over  $\mathbb{C}$  as follows (see e.g., [25]), which will be used in sections 5-6. Fix  $u_1, u_2 \in \mathbb{C}$  with  $u_1 - u_2 \in \mathbb{N}$ . Then  $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$  is called a *Kleshchev bipartition* [25] with respect to  $u_1, u_2$  if

$$\lambda_{u_1 - u_2 + i}^{(1)} \leq \lambda_i^{(2)} \text{ for all possible } i. \quad (3.15)$$

If  $u_1 - u_2 \notin \mathbb{Z}$ , all bipartitions of  $r$  are Kleshchev bipartitions. A pair of bipartitions  $(\mu, \nu)$  is *Kleshchev* if both  $\mu$  and  $\nu$  are Kleshchev bipartitions in the sense of (3.15) with respect to the parameters  $u_1, u_2$  and  $\bar{u}_1, \bar{u}_2$ . The following result will be used in section 6.

**Proposition 3.9.** *Suppose  $\mathcal{B}_{2,r,t}$  is admissible over  $\mathbb{C}$ . For each  $(f, \mu, \nu) \in \Lambda_{2,r,t}$ , let*

$$\tilde{C}(f, \mu, \nu) := \mathfrak{e}^f \mathfrak{x}_{\mu'} \bar{\mathfrak{x}}_{\nu'} w_{\mu'} w_{\nu'} \mathfrak{y}_{\mu} \bar{\mathfrak{y}}_{\nu} \mathcal{B}_{2,r,t} \pmod{\mathcal{B}_{2,r,t}^{f+1}},$$

where  $\mathcal{B}_{2,r,t}^{f+1}$  is the two-sided ideal of  $\mathcal{B}_{2,r,t}$  generated by  $\mathfrak{e}^{f+1}$ . Then  $C(f, \mu, \nu) \cong \tilde{C}(f, \mu, \nu)$ .

*Proof.* Let  $M_f$  be the left  $\mathcal{B}_{2,r-f,t-f}$ -module generated by

$$V_{r,t}^f = \{\mathfrak{e}^f dx^{\kappa_d} \mid (d, \kappa_d) \in \mathcal{D}_{r,t}^f \times \mathbf{N}_f\}. \quad (3.16)$$

By [19, Proposition 6.10],  $M_f = \mathfrak{e}^f \mathcal{B}_{2,r,t}$ . By [19, Lemma 6.9], one can use  $\mathcal{H}_{2,r-f} \otimes \mathcal{H}_{2,t-f}$  instead of  $\mathcal{B}_{2,r-f,t-f}$  in  $\mathfrak{x}_{\mu'} \bar{\mathfrak{x}}_{\nu'} w_{\mu'} w_{\nu'} \mathfrak{y}_{\mu} \bar{\mathfrak{y}}_{\nu} M_f \pmod{\mathcal{B}_{2,r,t}^{f+1}}$ . Now, the required isomorphism follows from Lemma 3.4 (5).  $\square$

#### 4. SUPER SCHUR-WEYL DUALITY

The aim of this section is to generalize super Schur-Weyl duality between general linear Lie superalgebra  $\mathfrak{gl}_{m|n}$  and  $\mathcal{B}_{2,r,t}$  to the case  $r+t > \min\{m, n\}$ . Throughout, let  $I_0 = \{1, \dots, m\}$ ,  $I_1 = \{m+1, \dots, m+n\}$  and  $I = I_0 \cup I_1$ .

For any pairs  $(i, j) \in I \times I$ , let  $E_{ij}$  be the matrix unit with parity  $[E_{ij}] = [i] + [j]$ , where  $[i] = a$  if  $i \in I_a$ ,  $a = 0, 1$ . The *general linear Lie superalgebra*  $\mathfrak{gl}_{m|n}$  over  $\mathbb{C}$ , denoted by  $\mathfrak{g}$ , is  $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ , where

$$\begin{aligned} \mathfrak{g}_{-1} &= \text{span}_{\mathbb{C}}\{E_{i,j} \mid i \in I_1, j \in I_0\}, \quad \mathfrak{g}_1 = \text{span}_{\mathbb{C}}\{E_{i,j} \mid i \in I_0, j \in I_1\}, \\ \mathfrak{g}_0 &= \text{span}_{\mathbb{C}}\{E_{i,j} \mid i, j \in I_0 \text{ or } i, j \in I_1\}. \end{aligned} \quad (4.1)$$

The Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is the  $\mathbb{C}$ -space with basis  $\{E_{ii} \mid i \in I\}$ . Let  $\mathfrak{h}^*$  be the dual space of  $\mathfrak{h}$  with dual basis  $\{\varepsilon_i \mid i \in I\}$ . Then any  $\xi \in \mathfrak{h}^*$ , called a *weight* of  $\mathfrak{g}$ , can be written as

$$\xi = \sum_{i \in I_0} \xi_i^L \varepsilon_i + \sum_{i \in I_1} \xi_{i-m}^R \varepsilon_i \text{ with } \xi_i^L, \xi_j^R \in \mathbb{C}. \quad (4.2)$$

Denote  $\xi$  by  $(\xi_1^L, \dots, \xi_m^L \mid \xi_1^R, \dots, \xi_n^R)$ . If both  $\xi_i^L - \xi_{i+1}^L \in \mathbb{N}$  and  $\xi_j^R - \xi_{j+1}^R \in \mathbb{N}$  for all possible  $i, j$ , then  $\xi$  is called *integral dominant*. Let  $P^+$  be the set of integral dominant weights. For any  $\xi \in P^+$ , let

$$\xi^\rho := \xi + \rho = (\xi_1^{L,\rho}, \dots, \xi_m^{L,\rho} \mid \xi_1^{R,\rho}, \dots, \xi_n^{R,\rho}), \quad (4.3)$$

where  $\rho = (0, -1, \dots, 1-m \mid m-1, m-2, \dots, m-n)$ . Following [14], let

$$\ell = \#\{(i, j) \mid \xi_i^{L,\rho} + \xi_j^{R,\rho} = 0, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then  $\xi$  is called an  $\ell$ -fold *atypical weight* if  $\ell > 0$ . Otherwise,  $\xi$  is called a *typical weight*.

**Example 4.1.** For any  $p, q \in \mathbb{C}$ , let  $\lambda_{pq} = (p, \dots, p \mid -q, \dots, -q)$ . Then  $\lambda_{pq}$  is a typical weight if and only if

$$p - q \notin \mathbb{Z} \text{ or } p - q \leq -m \text{ or } p - q \geq n. \quad (4.4)$$

The current  $q$  should be regarded as  $q + m$  in [5, IV]. In the remaining part of this paper,  $\lambda_{pq}$  is always a typical weight in the sense of (4.4).

Let  $V = \mathbb{C}^{m|n}$  be the natural  $\mathfrak{g}$ -module with natural basis  $\{v_i \mid i \in I\}$  such that  $v_i$  has parity  $[v_i] = [i]$ . Then the dual space  $V^*$ , which has the dual basis  $\{\bar{v}_i \mid i \in I\}$ , is a left  $\mathfrak{g}$ -module such that

$$E_{ab}\bar{v}_i = -(-1)^{[a]([a]+[b])}\delta_{ia}\bar{v}_b \text{ for any } (a, b) \in I \times I. \quad (4.5)$$

In particular, the weight of  $\bar{v}_i$  is  $-\epsilon_i$ . For the simplicity of notation, we set  $W = V^*$ .

**Definition 4.2.** Fix  $r, t \in \mathbb{Z}^{>0}$ . Let  $V^{rt} = V^{\otimes r} \otimes W^{\otimes t}$  and  $M_{pq}^{rt} = V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes W^{\otimes t}$ , where  $K_{\lambda_{pq}}$  is the Kac-module [14] with respect to the highest weight  $\lambda_{pq}$  in Example 4.1.

Let  $\pi : M_{pq}^{rt} \rightarrow V^{rt}$  be the projection such that, for any  $v \in M_{pq}^{rt}$ ,  $\pi(v)$  is the vector obtained from  $v$  by deleting the tensor factor in  $K_{\lambda_{pq}}$ . Let  $v_{pq}$  be the highest weight vector of  $K_{\lambda_{pq}}$  with highest weight  $\lambda_{pq}$ . Then  $v_{pq}$  is unique up to a scalar. It is well-known (e.g. see [5]) that  $K_{\lambda_{pq}}$  is  $2^{mn}$ -dimensional with a basis

$$B = \left\{ b^\sigma := \prod_{i=1}^n \prod_{j=1}^m E_{m+i,j}^{\sigma_{ij}} v_{pq} \mid \sigma = (\sigma_{ij})_{i,j=1}^{n,m} \in \{0, 1\}^{n \times m} \right\}, \quad (4.6)$$

where the products are taken in any fixed order. Define

$$\begin{aligned} I(m|n, r) &= \{\mathbf{i} \mid \mathbf{i} = (i_r, i_{r-1}, \dots, i_1), i_j \in I, 1 \leq j \leq r\}, \\ \bar{I}(m|n, t) &= \{\mathbf{j} \mid \mathbf{j} = (j_1, j_2, \dots, j_t), j_i \in I, 1 \leq i \leq t\}. \end{aligned} \quad (4.7)$$

If  $(\mathbf{i}, b, \mathbf{j}) \in I(m|n, r) \times B \times \bar{I}(m|n, t)$ , we define

$$v_{\mathbf{i}, b, \mathbf{j}} = v_{i_r} \otimes v_{i_{r-1}} \otimes \dots \otimes v_{i_1} \otimes b \otimes \bar{v}_{j_1} \otimes \bar{v}_{j_2} \otimes \dots \otimes \bar{v}_{j_t} \in M_{pq}^{rt}. \quad (4.8)$$

**Lemma 4.3.** Let  $B_M = \{v_{\mathbf{i}} \otimes b \otimes \bar{v}_{\mathbf{j}} \mid (\mathbf{i}, b, \mathbf{j}) \in I(m|n, r) \times B \times \bar{I}(m|n, t)\}$ . Then  $B_M$  is a basis of  $M_{pq}^{rt}$ .

Denote by  $U(\mathfrak{g})$  the universal enveloping algebra of  $\mathfrak{g}$ . Then  $M_{pq}^{rt}$  is a left  $U(\mathfrak{g})$ -module. Let  $J = J_1 \cup \{0\} \cup J_2$  with  $J_1 = \{r, \dots, 2, 1\}$  and  $J_2 = \{\bar{1}, \bar{2}, \dots, \bar{t}\}$ . Then  $(J, \prec)$  is a total ordered set with

$$r \prec r-1 \prec \dots \prec 1 \prec 0 \prec \bar{1} \prec \dots \prec \bar{t}.$$

For any  $a, b \in J$  with  $a \prec b$ , define  $\pi_{ab} : U(\mathfrak{g})^{\otimes 2} \rightarrow U(\mathfrak{g})^{\otimes (r+t+1)}$  by

$$\pi_{ab}(x \otimes y) = 1 \otimes \dots \otimes 1 \otimes x^{\text{a-th}} \otimes 1 \otimes \dots \otimes 1 \otimes y^{\text{b-th}} \otimes 1 \otimes \dots \otimes 1. \quad (4.9)$$

Let  $\Omega$  be a Casimir element in  $\mathfrak{g}^{\otimes 2}$  given by

$$\Omega = \sum_{i,j \in I} (-1)^{[j]} E_{ij} \otimes E_{ji}. \quad (4.10)$$

In [19], we define operators  $s_i, \bar{s}_j, x_1, \bar{x}_1$  and  $e_1$  acting on the right of  $M_{pq}^{rt}$  via the following formulae:

$$\begin{aligned} s_i &= \pi_{i+1,i}(\Omega)|_{M_{pq}^{rt}} \quad (1 \leq i < r), \quad \bar{s}_j = \pi_{\bar{j}, \bar{j}+1}(\Omega)|_{M_{pq}^{rt}} \quad (1 \leq j < t), \\ x_1 &= -\pi_{10}(\Omega)|_{M_{pq}^{rt}}, \quad \bar{x}_1 = -\pi_{0\bar{1}}(\Omega)|_{M_{pq}^{rt}}, \quad e_1 = -\pi_{1\bar{1}}(\Omega)|_{M_{pq}^{rt}}. \end{aligned} \quad (4.11)$$

Then there is an algebra homomorphism  $\phi : \mathcal{B}_{2,r,t} \rightarrow \text{End}_{U(\mathfrak{g})}(M_{pq}^{rt})$  sending the generators  $s_i, \bar{s}_j, x_1, \bar{x}_1$  and  $e_1$  to the operators  $s_i, \bar{s}_j, x_1, \bar{x}_1$  and  $e_1$  as above [19]. In this case, we need to use  $-p, m-q$ , and  $q, p-n$  instead of  $u_1, u_2, \bar{u}_1$  and  $\bar{u}_2$  respectively in Definition 2.1 for  $k=2$ . Further,  $\omega_0 = m-n$ ,  $\omega_1 = nq - mp$  and  $\omega_a = (m-p-q)\omega_{a-1} - p(q-m)\omega_{a-2}$  for  $a \geq 2$  and  $\bar{\omega}_a$ 's are determined by [19, Corollary 4.3]. Thus,  $\mathcal{B}_{2,r,t}$  is admissible in the sense of Definition 2.3. By Theorem 2.12,  $\dim_{\mathbb{C}} \mathcal{B}_{2,r,t} = 2^{r+t}(r+t)!$ . We will always consider  $\mathcal{B}_{2,r,t}$  as above in the remaining part of this paper.

**Theorem 4.4.** [19, Theorem 5.16] *Fix  $r, t \in \mathbb{Z}^{>0}$  with  $r+t \leq \min\{m, n\}$ . Then  $\text{End}_{\mathfrak{g}}(M_{pq}^{rt}) \cong \mathcal{B}_{2,r,t}$ .*

**Theorem 4.5.** [5, IV, Theorem 3.13] *If  $0 < r \leq \min\{m, n\}$ , then  $\text{End}_{U(\mathfrak{g})}(M_{pq}^{r0}) \cong \mathcal{H}_{2,r}$ , the level two Hecke algebra with defining parameters  $u_1 = -p$  and  $u_2 = m-q$ .*

**Theorem 4.6.** (Super Schur-Weyl duality) *Keep the condition (4.4). The algebra homomorphism  $\phi_1 : \mathcal{B}_{2,r,t} \rightarrow \text{End}_{\mathfrak{g}}(M_{pq}^{rt})$  is surjective. It is injective if and only if  $r+t \leq \min\{m, n\}$ .*

*Proof.* By Theorem 4.4, it suffices to prove that  $\phi_1$  is surjective and is not injective if  $r+t > \min\{m, n\}$ . As in [6, (7.16)], the map  $\text{flip}_{r,t}$  defined by the following commutative diagram is a  $\mathfrak{g}$ -module isomorphism

$$\begin{array}{ccc} \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t}) & \xrightarrow{\text{flip}_{r,t}} & \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t}) \\ \parallel & & \parallel \\ \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}}) \otimes \text{End}_{\mathbb{C}}((V^*)^{\otimes t}) & \xrightarrow{f \otimes g^* \mapsto f \otimes g} & \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}}) \otimes \text{End}_{\mathbb{C}}(V^{\otimes t}). \end{array} \quad (4.12)$$

Note that  $\mathcal{H}_{2,r+t}$  (denoted as  $H_{r+t}^{p,q}$  in [5, IV]) is a subspace of  $\text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})$ , thus (4.12) induces the following commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{2,r,t} & \xrightarrow{\text{flip}_{r,t}} & \mathcal{H}_{2,r+t} \\ \phi_1 \downarrow & & \downarrow \pi_1 \\ \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t}) & \xrightarrow{\text{flip}_{r,t}} & \text{End}_{\mathbb{C}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t}). \end{array} \quad (4.13)$$

By Theorem 2.12 for  $k=2$ ,  $\dim_{\mathbb{C}} \mathcal{B}_{2,r,t} = 2^{r+t}(r+t)!$ . This implies that the top map is a bijection, and the bottom map is a  $\mathfrak{g}$ -module isomorphism, which induces an isomorphism between two subspaces  $\text{End}_{\mathfrak{g}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t})$  and  $\text{End}_{\mathfrak{g}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})$ . Since  $\pi_1$  is surjectively mapped to  $\text{End}_{\mathfrak{g}}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})$  by [5, IV, Theorem 3.21], we see that  $\phi_1 : \mathcal{B}_{2,r,t} \rightarrow \text{End}_{\mathfrak{g}}(M_{pq}^{rt})$  is surjective. Finally, the second assertion follows from the corresponding result for  $t=0$  in [5, IV, Theorem 3.21].  $\square$

## 5. HIGHEST WEIGHT VECTORS IN $V^{\otimes r} \otimes K_{\lambda_{pq}}$

The aim of this section is to give a classification of highest weight vectors of  $M_{pq}^{r0} := V^{\otimes r} \otimes K_{\lambda_{pq}}$  when  $r \leq \min\{m, n\}$ , where  $V$  is the natural representation of  $\mathfrak{g} := \mathfrak{gl}_{m|n}$  and  $K_{\lambda_{pq}}$  is the Kac-module with highest weight  $\lambda_{pq}$  in Example 4.1. This will be done in a few steps. First, by noting that  $\mathfrak{g}$ -highest weight vectors of  $M_{pq}^{r0}$  is in one to one correspondence with the  $\mathfrak{g}_0$ -highest weight vectors of  $V^{\otimes r}$  (cf. [21, Lemmas 5.1-5.2]), we are able to



reduce the problem to the Lie algebra case. Secondly, since  $\mathfrak{g}_0 = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ , and  $V^{\otimes r}$  can be decomposed as a direct sum of tensor products of natural representations of  $\mathfrak{gl}_m$  and  $\mathfrak{gl}_n$ , we are able to further simplify the problem to the  $\mathfrak{gl}_m$  case.

To begin with, we briefly recall the results on a classification of  $\mathfrak{gl}_m$ -highest weight vectors of  $V^{\otimes r}$ , where  $V$  temporarily denotes the natural representation of  $\mathfrak{gl}_m$  over  $\mathbb{C}$ . Let  $\{v_i \mid 1 \leq i \leq m\}$  be a basis of  $V$ . Obviously,  $V^{\otimes r}$  has a basis  $\{v_{\mathbf{i}} \mid \mathbf{i} \in I(m|0, r)\}$ , where

$$v_{\mathbf{i}} = v_{i_r} \otimes v_{i_{r-1}} \otimes \cdots \otimes v_{i_1}.$$

We consider a Cashmir element  $\Omega$  in  $\mathfrak{gl}_m^{\otimes 2}$  with

$$\Omega = \sum_{1 \leq i, j \leq m} E_{ij} \otimes E_{ji} \in \mathfrak{gl}_m^{\otimes 2}, \quad (5.1)$$

which is a special case of (4.10). Define  $\mathbf{s}_i = \pi_{i, i+1}(\Omega)$ ,  $1 \leq i \leq r-1$ . Then  $(i, i+1) \in \mathfrak{S}_r$  acts on  $V^{\otimes r}$  via  $\mathbf{s}_i$ . Thus,  $V^{\otimes r}$  is a  $(\mathfrak{gl}_m, \mathbb{C}\mathfrak{S}_r)$ -bimodule such that

$$v_{\mathbf{i}} w = v_{i_{(r)w-1}} \otimes v_{i_{(r-1)w-1}} \otimes \cdots \otimes v_{i_{(1)w-1}} \text{ for any } w \in \mathfrak{S}_r. \quad (5.2)$$

For example,  $v_{i_3} \otimes v_{i_2} \otimes v_{i_1} s_1 s_2 = v_{i_1} \otimes v_{i_3} \otimes v_{i_2}$ . If  $r \leq m$ , it is well-known that

$$\text{End}_{U(\mathfrak{gl}_m)}(V^{\otimes r}) \cong \mathbb{C}\mathfrak{S}_r.$$

**Definition 5.1.** If  $\lambda \in \Lambda^+(r, m)$ , the set of partitions of  $r$  with at most  $m$  parts, we define  $v_\lambda = v_{\mathbf{i}_\lambda} \in V^{\otimes r}$ , where  $\mathbf{i}_\lambda = (1^{\lambda_1}, 2^{\lambda_2}, \dots, m^{\lambda_m})$  and  $k^{\lambda_k}$  denotes the sequence  $k, k, \dots, k$  with multiplicity  $\lambda_k$ .

The following result is well-known, and Lemma 5.3 follows from Lemma 5.2.

**Lemma 5.2.** Suppose  $\lambda$  and  $\mu$  are two compositions of  $r$  and  $\mu'$  is the conjugate of  $\mu$ , and  $x_\lambda, y_{\mu'}$  are defined in (3.8). Then  $x_\lambda \mathbb{C}\mathfrak{S}_r y_{\mu'} = 0$  unless  $\lambda \leq \mu$ .

**Lemma 5.3.** There is a bijection between the set of dominant weights of  $V^{\otimes r}$  and  $\Lambda^+(r, m)$ , the set of partitions of  $r$  with at most  $m$  parts. Further, the  $\mathbb{C}$ -space of  $\mathfrak{gl}_m$ -highest weight vectors with highest weight  $\lambda$  has a basis  $\{v_\lambda w_\lambda y_{\lambda'} d(\mathbf{t}) \mid \mathbf{t} \in \mathcal{T}^s(\lambda')\}$ .

Now, we turn to construct  $\mathfrak{g}$ -highest weight vectors of  $M_{pq}^{r0}$ . Since  $r \leq \min\{m, n\}$ , there is a bijection between the set of dominant weights of  $M_{pq}^{r0}$  and  $\Lambda_2^+(r)$ . Further, if  $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$ , the corresponding dominant weight of  $M_{pq}^{r0}$  is

$$\bar{\lambda} := \lambda_{pq} + \tilde{\lambda}, \quad (5.3)$$

where

$$\tilde{\lambda} = (\lambda_1^{(1)}, \dots, \lambda_m^{(1)} \mid \lambda_1^{(2)}, \dots, \lambda_n^{(2)}). \quad (5.4)$$

For instance, if  $\lambda = ((3, 1), (2, 1))$ , then  $\tilde{\lambda} = (3, 1, 0, \dots, 0 \mid 2, 1, 0, \dots, 0)$ . Recall that  $\Omega$  is a Casimir element in  $\mathfrak{g}^{\otimes 2}$  given in (4.10). Define operators  $s_i, x_1$  acting on the right of  $M_{pq}^{r0}$  via the following formulae:  $s_i = \pi_{i+1, i}(\Omega)$ ,  $1 \leq i \leq r-1$  and  $x_1 = -\pi_{10}(\Omega)$ . In this case,  $u_1 = -p$  and  $u_2 = m - q$ . We remind that Brundan-Stroppel [5] defined  $x_1$  via  $\pi_{10}(\Omega)$ . So, the current  $x_1$  is  $-x_1$  in [5]. Recall that  $v_{\mathbf{i}} \otimes v_{pq} = v_{i_r} \otimes \cdots \otimes v_{i_2} \otimes v_{i_1} \otimes v_{pq}$  for any  $\mathbf{i} \in I(m|n, r)$  (cf. (4.7)), and  $x'_k = x_k + L_k$  with  $L_k = \sum_{i=1}^{k-1} (i, k)$  (see Lemma 2.7).

**Lemma 5.4.** [5, Lemma 3.1] Suppose  $\mathbf{i} \in I(m|n, r)$ , and  $1 \leq k \leq r$ .

- (1)  $v_{\mathbf{i}} \otimes v_{pq} x'_k = -p v_{\mathbf{i}} \otimes v_{pq}$  if  $1 \leq i_k \leq m$ .
- (2)  $v_{\mathbf{i}} \otimes v_{pq} x'_k = -q v_{\mathbf{i}} \otimes v_{pq} + \sum_{j=1}^m (-1)^{\sum_{i=1}^{k-1} [i]_q} v_{\mathbf{j}} \otimes (E_{i_k, j} v_{pq})$  if  $m+1 \leq i_k \leq m+n$ , where  $\mathbf{j} \in I(m|n, r)$  which is obtained from  $\mathbf{i}$  by using  $j$  instead of  $i_k$  in  $\mathbf{i}$ . In particular, the weight of  $v_{\mathbf{j}}$  is strictly bigger than that of  $v_{\mathbf{i}}$ .

**Definition 5.5.** For  $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$ , define  $v_{\bar{\lambda}} = v_{\mathbf{i}}$  with  $\mathbf{i} = (\mathbf{i}_{\lambda^{(1)}}, \mathbf{i}_{\lambda^{(2)}}) \in I(m|n, r)$ .

For instance,  $v_{\bar{\lambda}} = v_{\mathbf{i}}$  if  $\lambda = ((3, 1), (2, 1))$ , where  $\mathbf{i} = (1^3, 2, (m+1)^2, m+2)$ .

**Definition 5.6.** For any  $\mathbf{t} \in \mathcal{T}^s(\lambda')$ , we define  $v_{\mathbf{t}} = v_{\bar{\lambda}} \otimes v_{pq} w_{\lambda} \eta_{\lambda'} d(\mathbf{t})$ , where  $\eta_{\lambda'}$  is given in Definition 3.2 (2).

**Theorem 5.7.** Suppose  $r \leq \min\{m, n\}$ . There is a bijection between the set of dominant weights of  $M_{pq}^{r0}$  and  $\Lambda_2^+(r)$ . Further, the  $\mathbb{C}$ -space  $V_{\bar{\lambda}}$  of  $\mathfrak{g}$ -highest weight vectors of  $M_{pq}^{r0}$  with highest weight  $\bar{\lambda}$  has a basis  $\{v_{\mathbf{t}} \mid \mathbf{t} \in \mathcal{T}^s(\lambda')\}$ .

*Proof.* The required bijection between  $\Lambda_2^+(r)$  and the set of dominant weights of  $M_{pq}^{r0}$  is the map sending  $\lambda$  to  $\bar{\lambda}$  defined in (5.3). We claim that each  $v_{\mathbf{t}}$  is killed by  $E_{m, m+1}$  and  $E_{i, j}$  with  $i < j$  and either  $i, j \in I_0$  or  $i, j \in I_1$ . Since  $M_{pq}^{r0}$  is  $(\mathfrak{g}, \mathcal{H}_{2, r})$ -bimodule, we need only consider the case  $d(\mathbf{t}) = 1$ . In this case,  $\mathbf{t} = \mathbf{t}'$ .

Denote  $|\lambda^{(1)}| = a$ . Recall that  $w_{\lambda^{(1)}} \in \mathfrak{S}_a$  and  $w_{\lambda^{(2)}} \in \mathfrak{S}_{r-a}$  such that  $\mathbf{t}^{\lambda^{(i)}} w_{\lambda^{(i)}} = \mathbf{t}_{\lambda^{(i)}}$  for  $i = 1, 2$ . Then

$$w_{\lambda} = w_{\lambda^{(1)}} w_{\lambda^{(2)}} w_a = w_a w_{\lambda^{(2)}} w_{\lambda^{(1)}}. \quad (5.5)$$

By (3.6) and (5.5),

$$v_{\mathbf{t}} = v_{\bar{\lambda}} \otimes v_{pq} w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}} w_a \pi_{r-a}(-p),$$

where  $\mu^{(i)}$  is the conjugate of  $\lambda^{(i)}$  for  $i = 1, 2$ . By Lemmas 5.2–5.3,  $v_{\mathbf{t}}$  is killed by  $E_{i, j}$  with  $i < j$  and either  $i, j \in I_0$  or  $i, j \in I_1$ . Since  $E_{m, m+1}$  acts on  $M_{pq}^{r0}$  via  $\sum_{i=1}^{r+1} 1^{\otimes i-1} \otimes E_{m, m+1} \otimes 1^{\otimes r+1-i}$ , we have  $E_{m, m+1} v_{\bar{\lambda}} \otimes v_{pq} = 0$  if  $v_{m+1}$  does not occur in  $v_{\bar{\lambda}}$ . Otherwise,  $\lambda^{(2)} \neq \emptyset$  and  $r-a \neq 0$ . In this case, up to a sign,  $E_{m, m+1} v_{\bar{\lambda}} \otimes v_{pq}$  is equal to

$$v_{\mathbf{j}} \otimes v_{pq} (1 - s_{a+1} + s_{a+1, a+3} + \cdots + (-1)^{b-a} s_{a+1, b+1}),$$

where  $b = a + \lambda_1^{(2)} - 1$  and  $v_{\mathbf{j}}$  is obtained from  $v_{\bar{\lambda}}$  by replacing  $v_{m+1}$  by  $v_m$  at the  $(a+1)$ -th position. Thus,  $j_{a+1} = m$ . Let

$$h = (1 - s_{a+1} + s_{a+1, a+3} + \cdots + (-1)^{b-a} s_{a+1, b+1}) w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}}.$$

Then  $h \in \mathbb{C}\mathfrak{S}_a \otimes \mathbb{C}\mathfrak{S}_{r-a}$ . By (3.6),  $h w_a = w_a h_1$  for some  $h_1 \in \mathbb{C}\mathfrak{S}_{r-a} \otimes \mathbb{C}\mathfrak{S}_a$ . Since  $h_1 \pi_{r-a}(-p) = \pi_{r-a}(-p) h_1$ , it is enough to prove  $v_{\mathbf{j}} \otimes v_{pq} w_a \pi_{r-a}(-p) = 0$ . Up to a sign,  $v_{\mathbf{j}} \otimes v_{pq} w_a = v_{\mathbf{k}} \otimes v_{pq}$  for some  $\mathbf{k}$  such that  $v_{k_1} = v_m \in V_0$ . Since  $r-a \neq 0$ ,  $x_1 + p$  is a factor of  $\pi_{r-a}(-p)$ . By Lemma 5.4 (1),  $v_{\mathbf{j}} \otimes v_{pq} w_a \pi_{r-a}(-p) = 0$ . Thus,  $v_{\mathbf{t}}$  is a highest weight vector of  $M_{pq}^{r0}$  if  $v_{\mathbf{t}} \neq 0$ .

Note that any vector of  $M_{pq}^{r0}$  can be written as  $v = \sum_{b \in B} v_b \otimes b$ , where  $B$  is a basis of  $K_{\lambda_{pq}}$  defined in (4.6) and  $v_b \in V^{\otimes r}$ . Following [5],  $v_b$  is called the  $b$ -component of  $v$ . By Lemma 5.4 (2) (or the arguments in the proof of [6, Corollary 3.3]), the  $v_{pq}$ -component of

$v_{\tilde{\lambda}} \otimes v_{pq} w_a \pi_{r-a}(-p)$  is  $v_{\tilde{\lambda}} w_a \prod_{i=1}^{r-a} (p-q-L_i)$ . By Lemma 2.7 (3),  $\prod_{i=1}^{r-a} (p-q-L_i)$  is a central element in  $\mathbb{C}\mathfrak{S}_{r-a}$ , which acts on  $v_{\tilde{\lambda}} \otimes v_{pq} w_{\lambda(2)} x_{\mu(2)}$  as scalar  $\prod_{i=1}^{r-a} (p-q-\text{res}_{\mu(2)}(i))$ , where  $\mu = \lambda'$  and  $\text{res}_{\mu(2)}(i)$  is  $j-l$  if  $i$  is in the  $l$ -th row and  $j$ -th column of  $\mathfrak{t}^{\mu(2)}$ . Since  $\lambda_{pq}$  is typical (cf. (4.4)), and  $r \leq \min\{m, n\}$ ,  $\prod_{i=1}^{r-a} (p-q-\text{res}_{\mu(2)}(i)) \neq 0$ . So,

$$\text{the } v_{pq}\text{-component of } v_{\mathfrak{t}} = v_{\tilde{\lambda}} w_a w_{\lambda(2)} x_{\mu(2)} w_{\lambda(1)} y_{\mu(1)} d(\mathfrak{t}) \text{ (up to a non-zero scalar).} \quad (5.6)$$

By Lemma 5.3, it is a  $\mathfrak{g}_0$ -highest vector of  $V^{\otimes r}$  with highest weight  $\tilde{\lambda}$  (cf. (5.4)), forcing  $v_{\mathfrak{t}} \neq 0$ .

Now, we prove that  $\{v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$  is  $\mathbb{C}$ -linear independent. First, consider  $V = V_0 \oplus V_1$  as a module for  $\mathfrak{g}_0 = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ . Then  $V^{\otimes r}$  can be decomposed as a direct sum of  $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_r}$ , where  $i_j \in \{0, 1\}$ . As  $\mathfrak{g}_0$ -modules,  $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_r} \cong V_1^{\otimes r-a} \otimes V_0^a$  for some non-negative integer  $a \leq r$  with  $a = \#\{i_j \mid i_j = 0\}$ . The corresponding isomorphism is given by acting a unique element  $w$  on the right hand side of  $V_1^{\otimes r-a} \otimes V_0^a$ , where  $w$  is a distinguished right coset representatives of  $\mathfrak{S}_a \times \mathfrak{S}_{r-a}$  in  $\mathfrak{S}_r$ . By Lemma 5.3, all  $\mathfrak{g}_0$ -highest weight vectors of  $V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_r}$  with highest weight  $\tilde{\lambda}$  are  $v_{\tilde{\lambda}} w_{\lambda(1)} y_{\mu(1)} w_{\lambda(2)} x_{\mu(2)} d(\mathfrak{t}_1) d(\mathfrak{t}_2) w$  for all  $\mathfrak{t}_1 \in \mathcal{T}^s(\mu^{(1)})$  and  $\mathfrak{t}_2 \in \mathcal{T}^s(\mu^{(2)})$ . Therefore, the  $\mathbb{C}$ -space  $V_{\tilde{\lambda}}$  of all  $\mathfrak{g}_0$ -highest weight vectors of  $V^{\otimes r}$  with highest weight  $\tilde{\lambda}$  has a basis  $\{v_{\tilde{\lambda}} w_{\lambda(2)} x_{\mu(2)} w_{\lambda(1)} y_{\mu(1)} d(\mathfrak{t}) \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$ , where  $\mu = \lambda'$ . By (5.6),  $\{v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$  is  $\mathbb{C}$ -linear independent. Finally, since there is a one to one correspondence between  $\mathfrak{g}$ -highest weight vectors of  $M_{pq}^{r0}$  and  $\mathfrak{g}_0$ -highest weight vectors of  $V^{\otimes r}$  (cf. [21, Lemmas 5.1-5.2]), and  $\dim V_{\tilde{\lambda}} = \#\{v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$ , one obtains that  $\{v_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$  is a basis of  $V_{\tilde{\lambda}}$ .  $\square$

In the remaining part of this section, we want to establish the relationship between  $V_{\tilde{\lambda}}$  with a special cell module of  $\mathcal{H}_{2,r}$  with respect to  $\lambda \in \Lambda_2^+(r)$ . This result will be needed in section 6. We go on using  $-x_1$  instead of  $x_1$  in [5]. In this case, the current  $-p$  and  $m-q$  are the same as  $p$  and  $q$  in [5].

**Proposition 5.8.** *For any  $\lambda \in \Lambda_2^+(r)$ ,  $V_{\tilde{\lambda}} \cong \mathfrak{r}_{\lambda} w_{\lambda} \mathfrak{h}_{\lambda'} \mathcal{H}_{2,r}$  as right  $\mathcal{H}_{2,r}$ -modules, where  $V_{\tilde{\lambda}}$  is defined in Theorem 5.7.*

*Proof.* By Lemma 3.4 (2),  $S^{\lambda} := \mathfrak{r}_{\lambda} w_{\lambda} \mathfrak{h}_{\lambda'} \mathcal{H}_{2,r}$  has a basis  $M = \{\mathfrak{r}_{\lambda} w_{\lambda} \mathfrak{h}_{\lambda'} d(\mathfrak{t}) \mid \mathfrak{t} \in \mathcal{T}^s(\lambda')\}$ . It follows from Theorem 5.7 that there is a linear isomorphism  $\phi : V_{\tilde{\lambda}} \rightarrow S^{\lambda}$  sending  $v_{\mathfrak{t}}$  to  $\mathfrak{r}_{\lambda} w_{\lambda} \mathfrak{h}_{\lambda'} d(\mathfrak{t})$ . Obviously,  $\phi$  is a right  $\mathfrak{S}_r$ -homomorphism. In order to show that  $\phi$  is a right  $\mathcal{H}_{2,r}$ -homomorphism, it suffices to prove that

$$\phi(v_{\mathfrak{t}} x_k) = \phi(v_{\mathfrak{t}}) x_k \text{ for } 1 \leq k \leq r. \quad (5.7)$$

Denote  $a = |\lambda^{(1)}|$ . If  $1 \leq k \leq r-a$ , then  $\tilde{\pi}_{\lambda'} x_k = \pi_{r-a}(m-q)x_k = \pi_{r-a}(m-q)(-p-L_k)$ . Since  $\phi$  is a right  $\mathfrak{S}_r$ -homomorphism, (5.7) holds for  $1 \leq k \leq r-a$ . If  $r-a+1 \leq k \leq r$ , then  $x_k = s_{k,r-a+1} s_{r-a+1,k} - \sum_{j=r-a+1}^{k-1} (j, k)$ . By Lemma 3.4 (1),

$$\pi_{\lambda} w_a \tilde{\pi}_{\lambda'} x_k = \pi_{\lambda} w_a \tilde{\pi}_{\lambda'} \left( -p - \sum_{j=r-a+1}^{k-1} (j, k) \right). \quad (5.8)$$

On the other hand,  $\tilde{\pi}_{\lambda'} x_k = x_k \tilde{\pi}_{\lambda'}$  and  $v_{\tilde{\lambda}} \otimes v_{pq} w_{\lambda(1)} w_{\lambda(2)} y_{\mu(1)} x_{\mu(2)} w_a$  is a linear combination of elements  $v_{\mathfrak{i}} \otimes v_{pq}$ , for some  $\mathfrak{i} \in I(m|n, r)$  such that  $v_{i_j} \in V_0$  for all  $r-a+1 \leq j \leq r$ . By

Lemma 5.4 (1),  $x_k$  acts on  $v_i \otimes v_{pq}$  as  $-p - L_k$ . In order to verify (5.7) for  $k \geq r - a + 1$ , by (5.8), it remains to show that

$$v_i \otimes v_{pq}(i, k)\tilde{\pi}_{\lambda'} = 0 \quad \text{for all } i, 1 \leq i \leq r - a. \quad (5.9)$$

Write  $v_i \otimes v_{pq}(i, k) = v_j$  up to a sign. Then  $v_j \in V_0$  and  $v_j(1, i)\tilde{\pi}_{\lambda'} = 0$  by Lemma 5.4 (1). Since  $(1, i)\tilde{\pi}_{\lambda'} = \tilde{\pi}_{\lambda'}(1, i)$ , and  $(1, i)$  is invertible,  $v_j\tilde{\pi}_{\lambda'} = 0$ , proving (5.9).  $\square$

**Corollary 5.9.** *Suppose  $\lambda \in \Lambda_2^+(r)$ . As right  $\mathcal{H}_{2,r}$ -modules,*

$$\text{Hom}_{U(\mathfrak{g})}(K_{\bar{\lambda}}, M_{pq}^{r0}) \cong \tilde{\Delta}(\lambda') \quad (5.10)$$

where  $\tilde{\Delta}(\lambda')$  is the right cell module defined via the cellular basis of  $\mathcal{H}_{2,r}$  in Lemma 3.3 (2).

*Proof.* For any  $\mathfrak{g}$ -highest weight vector  $v$  of  $M_{pq}^{r0}$  with highest weight  $\bar{\lambda}$ , there is a unique  $U(\mathfrak{g})$ -homomorphism  $f_v : K_{\bar{\lambda}} \rightarrow U(\mathfrak{g})v \subset M_{pq}^{r0}$  sending  $v_{\bar{\lambda}}$  to  $v$ , where  $v_{\bar{\lambda}}$  is the highest weight vector of  $K_{\bar{\lambda}}$ . Further  $f_v$  can be considered as a homomorphism in  $\text{Hom}_{U(\mathfrak{g})}(K_{\bar{\lambda}}, M_{pq}^{r0})$  by composing an embedding homomorphism.

For any  $0 \neq f \in \text{Hom}_{U(\mathfrak{g})}(K_{\bar{\lambda}}, M_{pq}^{r0})$ ,  $f(v_{\bar{\lambda}})$  is a highest weight vector of  $M_{pq}^{r0}$ . By Theorem 5.7,  $f(v_{\bar{\lambda}})$  is a linear combination of  $v_t$ 's, for  $t \in \mathcal{T}^s(\lambda')$ . So,  $f$  can be written as a linear combination of  $f_{v_t}$ 's. Thus,  $\{f_{v_t} \mid t \in \mathcal{T}^s(\lambda')\}$  is a basis of  $\text{Hom}_{U(\mathfrak{g})}(K_{\bar{\lambda}}, M_{pq}^{r0})$ . Let  $V_{\bar{\lambda}}$  be defined in Theorem 5.7. Then the linear isomorphism  $\phi : \text{Hom}_{U(\mathfrak{g})}(K_{\bar{\lambda}}, M_{pq}^{r0}) \rightarrow V_{\bar{\lambda}}$  sending  $f_{v_t}$  to  $v_t$  for any  $t \in \mathcal{T}^s(\lambda')$  is a right  $\mathcal{H}_{2,r}$ -homomorphism. By Lemma 3.4 (5) and Proposition 5.8,  $V_{\bar{\lambda}} \cong \tilde{\Delta}(\lambda')$ , proving (5.10).  $\square$

In the remaining part of this section, we always assume  $p - q \leq -m$ . If  $p - q \geq n$ , one can switch roles between  $p$  and  $q$  (or consider the dual module of  $M_{pq}^{r0}$ ). Without loss of any generality, we assume  $p, q \in \mathbb{Z}$ .

Let  $\lambda \in \Lambda_2^+(r)$  with  $r \leq \min\{m, n\}$ . Then  $\lambda$  corresponds to a dominant weight  $\bar{\lambda}$  defined in (5.3). In particular,  $\bar{\emptyset} = \lambda_{pq}$ . Following [5, 13, 22], we are going to represent a dominant weight  $\bar{\lambda}$  in a unique way by a weight diagram  $D_{\lambda}$ . First we write (cf. (4.3))

$$\bar{\lambda}^{\rho} = \bar{\lambda} + \rho = (\bar{\lambda}_1^{L,\rho}, \dots, \bar{\lambda}_m^{L,\rho} \mid \bar{\lambda}_1^{R,\rho}, \dots, \bar{\lambda}_n^{R,\rho}). \quad (5.11)$$

Denote

$$\begin{aligned} S(\lambda)_L &= \{\bar{\lambda}_i^{L,\rho} \mid i = 1, \dots, m\}, & S(\lambda)_R &= \{-\bar{\lambda}_j^{R,\rho} \mid j = 1, \dots, n\}, \\ S(\lambda) &= S(\lambda)_L \cup S(\lambda)_R, & S(\lambda)_B &= S(\lambda)_L \cap S(\lambda)_R. \end{aligned}$$

**Definition 5.10.** The *weight diagram*  $D_{\lambda}$  associated with the dominant weight  $\bar{\lambda}$  is a line with vertices indexed by  $\mathbb{Z}$  such that each vertex  $i$  is associated with a symbol  $D_{\lambda}^i = \emptyset, <, >$  or  $\times$  according to whether  $i \notin S(\lambda)$ ,  $i \in S(\lambda)_R \setminus S(\lambda)_B$ ,  $i \in S(\lambda)_L \setminus S(\lambda)_B$  or  $i \in S(\lambda)_B$ .

For example, if  $p, q \in \mathbb{Z}$  with  $p \leq q - m$ , then the weight diagram  $D_{\bar{\emptyset}}$  of  $\bar{\emptyset} = \lambda_{pq}$  is given by

$$\cdots \xrightarrow{\quad} \underset{p-m+1}{\circ} \xrightarrow{\quad} \underset{p}{\circ} \xrightarrow{\quad} \cdots \xrightarrow{\quad} \underset{q-m+1}{\circ} \xrightarrow{\quad} \cdots \xrightarrow{\quad} \underset{q-m+n}{\circ} \xrightarrow{\quad} \cdots, \quad (5.12)$$

where, for simplicity, we have associated vertex  $i$  with nothing if  $D_{\lambda}^i = \emptyset$ . Note that  $\sharp S(\emptyset)_B = 0$ , i.e.,  $\lambda_{pq}$  is typical.

**Definition 5.11.** Let  $\bar{\lambda}$  be as in (5.3), where  $\lambda \in \Lambda_2^+(r)$ .

- (1) Let  $\bar{\lambda}^{\text{top}}$  be the unique dominant weight such that  $L_{\bar{\lambda}}$  is the simple submodule of the Kac-module  $K_{\bar{\lambda}^{\text{top}}}$ . Then  $\bar{\lambda}^{\text{top}}$  is obtained from  $\bar{\lambda}$  via the unique longest right path (cf. [22, Definition 5.2], [24, Conjecture 4.4]) or via a raising operator (cf. [4]). For example, if  $D_{\lambda}$  is given by

$$\cdots - 0 - \overset{\times}{1} - \overset{\times}{2} - 3 - \overset{\times}{4} - \overset{>}{5} - 6 - \overset{\times}{7} - \overset{<}{8} - 9 - \overset{<}{10} - 11 - \cdots, \quad (5.13)$$

then the weight diagram  $D_{\lambda^{\text{top}}}$  of  $\bar{\lambda}^{\text{top}}$  is given by

$$\cdots - 0 - 1 - 2 - \overset{\times}{3} - 4 - \overset{>}{5} - \overset{\times}{6} - 7 - \overset{<}{8} - \overset{\times}{9} - \overset{<}{10} - \overset{\times}{11} - \cdots, \quad (5.14)$$

where the  $\times$ 's at vertices 9, 6, 3, 11 in (5.14) are respectively obtained from the  $\times$ 's at vertices 7, 4, 2, 1 in (5.13) (thus every symbol “ $\times$ ” is always moved to the unique empty place at its right side which is closest to it, under the rule that the rightmost “ $\times$ ” should be moved first, as indicated in (5.14)). Alternatively,  $\bar{\lambda}$  is obtained from  $\bar{\lambda}^{\text{top}}$  via the unique longest left path.

- (2) Let  $\lambda^{\text{top}} \in \Lambda_2^+(r)$  be the unique bipartition such that  $\bar{\lambda}^{\text{top}} = \lambda_{pq} + \tilde{\lambda}^{\text{top}}$  (cf. (5.4) and (5.3)).

Write  $p = q - m - k$  for some  $k \in \mathbb{N}$ . If  $\mu = ((\mu_1^L, \dots, \mu_m^L), (\mu_1^R, \dots, \mu_n^R)) \in \Lambda_2^+(r)$ , then  $\mu'$  is Kleshchev with respect to  $u_1 = -p$ ,  $u_2 = m - q$  (cf. (3.15)) if and only if

$$\mu_i^L \geq \mu_i^R - k \quad \text{for all possible } i. \quad (5.15)$$

Following [5, IV], we denote  $I_{pq}^+ = \{p - m + 1, p - m + 2, \dots, q - m + n\}$ . For any  $\lambda \in \Lambda_2^+(r)$  and any  $j \in I_{pq}^+$ , set

$$I_{\geq j}^{\emptyset}(\lambda) = \mathbb{Z}^{\geq j} \cap (I_{pq}^+ \setminus S(\lambda) \cap I_{pq}^+), \quad (5.16)$$

$$I_{\leq j}^{\emptyset}(\lambda) = \mathbb{Z}^{\leq j} \cap (I_{pq}^+ \setminus S(\lambda) \cap I_{pq}^+), \quad (5.17)$$

$$I_{\geq j}^{\times}(\lambda) = \mathbb{Z}^{\geq j} \cap (I_{pq}^+ \cap S(\lambda)_{\text{B}}), \quad (5.18)$$

$$I_{\leq j}^{\times}(\lambda) = \mathbb{Z}^{\leq j} \cap (I_{pq}^+ \cap S(\lambda)_{\text{B}}). \quad (5.19)$$

In terms of the above notations, Brundan and Stroppel [5, IV, Lemma 2.6] have proved that the indecomposable tilting module  $T_{\bar{\lambda}}$  is a direct summand of  $M_{pq}^{r0}$  if

$$S(\lambda) \subset I_{pq}^+ \quad \text{and} \quad \#I_{\geq j}^{\emptyset}(\lambda) \geq \#I_{\geq j}^{\times}(\lambda) \quad \text{for all } j \in I_{pq}^+. \quad (5.20)$$

These two conditions on bipartition  $\lambda$  (or weight  $\bar{\lambda}$ ) are equivalent to the following conditions on  $\lambda^{\text{top}}$  (which can be seen from (5.13)–(5.14) in case  $I_{pq}^+ = \{1, 2, \dots, 11\}$ ):

$$S(\lambda^{\text{top}}) \subset I_{pq}^+ \quad \text{and} \quad \#I_{\leq j}^{\emptyset}(\lambda^{\text{top}}) \geq \#I_{\leq j}^{\times}(\lambda^{\text{top}}) \quad \text{for all } j \in I_{pq}^+. \quad (5.21)$$

**Lemma 5.12.** Let  $\mu \in \Lambda_2^+(r)$  such that  $\mu'$  is Kleshchev with respect to  $u_1 = -p$ ,  $u_2 = m - q$ , where  $p = q - m - k$  with  $k \in \mathbb{N}$ . Then

$$S(\mu) \subset I_{pq}^+ \quad \text{and} \quad \#I_{\leq j}^{\emptyset}(\mu) \geq \#I_{\leq j}^{\times}(\mu) \quad \text{for all } j \in I_{pq}^+. \quad (5.22)$$

*Proof.* We have (cf. (4.3))

$$\lambda_{pq} + \rho = (q - m - k, \dots, q - 2m - k + 1 \mid -q + m - 1, \dots, -q + m - n). \quad (5.23)$$

Thus for  $i = 1, \dots, m$ , we have (cf. (5.11))  $\bar{\mu}_i^{L,\rho} = \mu_i + q - m - k \geq q - 2m - k + 1$  and  $\bar{\mu}_i^{L,\rho} \leq q + n - m$  (as  $\mu_i \leq r \leq n$ ), i.e.,  $\bar{\mu}_i^{L,\rho} \in I_{pq}^+$ . Similarly,  $-\bar{\mu}_j^{R,\rho} \in I_{pq}^+$  for  $j = 1, \dots, n$ . Hence,  $S(\mu) \subset I_{pq}^+$ .

To prove the other assertion of (5.22), note that the weight diagram  $D_\mu$  of  $\bar{\mu}$  is obtained from  $D_\emptyset$  (cf. (5.12)) by moving the “ $>$ ” at vertex  $p - i$  for all  $i$  with  $0 \leq i \leq m - 1$  to its right side to vertex  $p - i + \mu_{i+1}^L$ , and moving the “ $<$ ” at vertex  $q - m + j$  for all  $j$  with  $1 \leq j \leq n$  to its left side to vertex  $q - m + j - \mu_j^R$  (if “ $<$ ” meets “ $>$ ” at the destination vertex, then two symbols “ $<$ ” and “ $>$ ” are combined to become the symbol “ $\times$ ”). Since  $\mu'$  is Kleshchev, condition (5.15) shows that in order to produce a “ $\times$ ” at some vertex  $i$  of  $D_\mu$ , a “ $>$ ” at some vertex  $j$  with  $j < i$  must be moved to vertex  $i$ , i.e., an “ $\emptyset$ ” must appear in some vertex  $j'$  with  $j' \leq j < i$ , i.e., (5.21) holds.  $\square$

**Corollary 5.13.** *Suppose  $\lambda \in \Lambda_2^+(r)$  such that  $(\lambda^{\text{top}})'$  is Kleshchev, where  $(\lambda^{\text{top}})'$  is the conjugate of  $\lambda^{\text{top}} \in \Lambda_2^+(r)$ . Then  $T_{\bar{\lambda}}$  is a direct summand of  $M_{pq}^{r0}$ . Further, any indecomposable direct summand of  $M_{pq}^{r0}$  is of form  $T_{\bar{\lambda}}$  for some  $\lambda \in \Lambda_2^+(r)$  such that  $(\lambda^{\text{top}})'$  is Kleshchev.*

*Proof.* The first assertion follows from [5, IV, Lemma 2.6] and Lemma 5.12. To prove the last assertion, since  $r \leq \min\{m, n\}$ , by Theorem 4.5,  $\text{End}_{U(\mathfrak{gl}_{m|n})}(M_{pq}^{r0}) \cong \mathcal{H}_{2,r}$ . So, the number of non-isomorphic indecomposable direct summands of  $\mathfrak{gl}_{m|n}$ -module  $M_{pq}^{r0}$  is equal to that of non-isomorphic irreducible  $\mathcal{H}_{2,r}$ -modules, which is equal to the number of Kleshchev bipartitions in  $\Lambda_2^+(r)$ . Now, everything is clear.  $\square$

**Corollary 5.14.** *Suppose  $\lambda \in \Lambda_2^+(r)$  such that  $(\lambda^{\text{top}})'$  is Kleshchev. As right  $\mathcal{H}_{2,r}$ -modules,*

$$\text{Hom}_{U(\mathfrak{g})}(T_{\bar{\lambda}}, M_{pq}^{r0}) \cong P^{(\lambda^{\text{top}})'}, \quad (5.24)$$

where  $P^{(\lambda^{\text{top}})'}$  is the projective cover of  $D^{(\lambda^{\text{top}})'}$  which is the simple head of  $\tilde{\Delta}((\lambda^{\text{top}})')$ .

*Proof.* Since  $r \leq \min\{m, n\}$  and  $(\lambda^{\text{top}})'$  is Kleshchev, by Corollary 5.13,  $T_{\bar{\lambda}}$  is a direct summand of  $M_{pq}^{r0}$ , forcing  $0 \neq \text{Hom}_{U(\mathfrak{g})}(T_{\bar{\lambda}}, M_{pq}^{r0})$  to be a direct summand of  $\mathcal{H}_{2,r}$ . We claim that  $\text{Hom}_{U(\mathfrak{g})}(T_{\bar{\lambda}}, M_{pq}^{r0})$  is indecomposable. If not, then the number of indecomposable direct summands of the right  $\mathcal{H}_{2,r}$ -module  $\mathcal{H}_{2,r}$  is strictly bigger than  $\sum_{\bar{\lambda}} \ell_{\bar{\lambda}}$  if we write  $M_{pq}^{r0}$  as  $M_{pq}^{r0} = \oplus_{\bar{\lambda}} T_{\bar{\lambda}}^{\oplus \ell_{\bar{\lambda}}}$  with  $\ell_{\bar{\lambda}} \neq 0$ .

On the other hand, since  $M_{pq}^{r0}$  is a right  $\mathcal{H}_{2,r}$ -module, we can consider the right exact functor  $\mathfrak{F} := M_{pq}^{r0} \otimes_{\mathcal{H}_{2,r}} ?$  from the category of left  $\mathcal{H}_{2,r}$ -modules to the category of left  $U(\mathfrak{g})$ -modules. We have an epimorphism from  $\mathfrak{F}(P^\mu)$  to  $\mathfrak{F}(\tilde{\Delta}(\mu))$ , where  $P^\mu$  is any principal indecomposable left  $\mathcal{H}_{2,r}$ -module and  $\tilde{\Delta}(\mu)$  temporally denotes the left cell module of  $\mathcal{H}_{2,r}$  defined via the cellular basis of  $\mathcal{H}_{2,r}$  in Lemma 3.3 (1) with the simple head  $D^\mu$ . By Lemma 3.4 (5) and Theorem 4.5,  $\mathfrak{F}(\tilde{\Delta}(\mu)) \neq 0$ , forcing  $\mathfrak{F}(P^\mu) \neq 0$ . So,  $\mathfrak{F}(P^\mu)$  is a direct sum of indecomposable direct summand of  $U(\mathfrak{g})$ -module  $M_{pq}^{r0}$ . In particular,  $\sum_{\bar{\lambda}} \ell_{\bar{\lambda}}$  is no less than the number of indecomposable direct summands of left  $\mathcal{H}_{2,r}$ -module  $\mathcal{H}_{2,r}$ . This is a contradiction since the

number of indecomposable direct summands of left  $\mathcal{H}_{2,r}$ -module  $\mathcal{H}_{2,r}$  is equal to that of indecomposable direct summands of right  $\mathcal{H}_{2,r}$ -module  $\mathcal{H}_{2,r}$ . So,  $\mathfrak{F}(T_{\bar{\lambda}})$  is a principal indecomposable right  $\mathcal{H}_{2,r}$ -module. Since  $K_{\lambda^{\text{top}}} \hookrightarrow T_{\bar{\lambda}}$ ,  $\text{Hom}_{U(\mathfrak{g})}(T_{\bar{\lambda}}, M_{pq}^{r0}) \twoheadrightarrow \text{Hom}_{U(\mathfrak{g})}(K_{\lambda^{\text{top}}}, M_{pq}^{r0})$ . By Corollary 5.9,  $\text{Hom}_{U(\mathfrak{g})}(K_{\lambda^{\text{top}}}, M_{pq}^{r0}) \cong \tilde{\Delta}((\lambda^{\text{top}})')$ . Since  $\text{Hom}_{U(\mathfrak{g})}(T_{\bar{\lambda}}, M_{pq}^{r0})$  is a principal indecomposable right  $\mathcal{H}_{2,r}$ -module, it implies that  $\tilde{\Delta}((\lambda^{\text{top}})')$  has the unique simple head, denoted by  $D^{(\lambda^{\text{top}})'}$ . Thus,  $\text{Hom}_{\mathfrak{g}}(T_{\bar{\lambda}}, M_{pq}^{r0}) \cong P^{(\lambda^{\text{top}})'}$ .  $\square$

Brundan-Stroppel have already proved that decomposition numbers of  $\mathcal{H}_{2,r}$  arising from super Schur–Weyl duality in [5] can be determined by the multiplicity of Kac-modules in indecomposable tilting modules appearing in  $M_{pq}^{r0}$ . This result can also be seen via the exact functor  $\text{Hom}_{U(\mathfrak{g})}(\cdot, M_{pq}^{r0})$ .

## 6. HIGHEST WEIGHT VECTORS IN $M_{pq}^{rt}$

In this section, we classify  $\mathfrak{g}$ -highest weight vectors of  $\mathfrak{gl}_{m|n}$ -module  $M_{pq}^{rt}$  over  $\mathbb{C}$ . As an application, we set up explicit relationship between Kac (resp. indecomposable tilting) modules of  $\mathfrak{g}$  and cell (resp. principal indecomposable) modules of  $\mathcal{B}_{2,r,t}$ . This gives us an efficient way to calculate decomposition numbers of  $\mathcal{B}_{2,r,t}$ . Throughout, assume  $r, t \in \mathbb{Z}^{>0}$  such that  $r + t \leq \min\{m, n\}$ . The case  $t = 0$  has been dealt with in section 5. By symmetry, one can also classify highest weight vectors of  $M_{pq}^{0t}$  via those in section 5. The following result, which is the counterpart of Lemma 5.4, can be verified directly.

**Lemma 6.1.** *Suppose  $\mathbf{i} \in I(m|n, r)$ ,  $\mathbf{j} \in \bar{I}(m|n, t)$  (cf. (4.7)) and  $1 \leq k \leq t$ .*

- (1)  $v_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}} \bar{x}'_k = qv_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}}$  if  $1 + m \leq j_k \leq m + n$ .
- (2)  $v_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}} \bar{x}'_k = pv_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}} + \sum_{j=m+1}^{m+n} (-1)^{\sum_{i=1}^{k-1} [ji]} v_{\mathbf{i}} \otimes (E_{jj_k} v_{pq}) \otimes \bar{v}_{\ell}$  if  $1 \leq j_k \leq m$ , where  $\ell \in \bar{I}(m|n, t)$  which is obtained from  $\mathbf{j}$  by using  $j$  instead of  $j_k$  in  $\mathbf{j}$ . In particular, the weight of  $\bar{v}_{\ell}$  is strictly bigger than that of  $\bar{v}_{\mathbf{j}}$ .

For any integral weight  $\xi = (\xi_1, \dots, \xi_m \mid \xi_{m+1}, \dots, \xi_{m+n})$  of  $\mathfrak{g}$ , let

$$\xi^L = (\xi_1^L, \dots, \xi_m^L) = (\xi_1, \dots, \xi_m), \text{ and } \xi^R = (\xi_1^R, \dots, \xi_m^R) = (\xi_{m+1}, \dots, \xi_{m+n}).$$

We define two bicompositions  $\mu, \nu$  such that all  $\mu_i^{(1)}, \mu_j^{(2)}, \nu_i^{(1)}, \nu_j^{(2)}$  are zero except that

- (1) for  $1 \leq i \leq m$ ,  $\mu_i^{(1)} = \xi_i^L$  if  $\xi_i^L > 0$  or  $\nu_{m-i+1}^{(1)} = -\xi_i^L$  if  $\xi_i^L < 0$ .
- (2) for  $1 \leq j \leq n$ ,  $\mu_j^{(2)} = \xi_j^R$  if  $\xi_j^R > 0$  or  $\nu_{n-j+1}^{(2)} = -\xi_j^R$  if  $\xi_j^R < 0$ .

Then both  $\mu$  and  $\nu$  correspond to integral weights of  $\mathfrak{g}$ . In particular,  $\xi = \mu - \hat{\nu}$  with

$$\hat{\nu} = (\nu_m^{(1)}, \dots, \nu_1^{(1)} \mid \nu_n^{(2)}, \dots, \nu_1^{(2)}) \in \mathfrak{h}^*. \quad (6.1)$$

Conversely, if  $\mu$  and  $\nu$  are two bicompositions, then  $\xi = \mu - \hat{\nu}$  is a integral weight of  $\mathfrak{g}$ . For instance, if  $\xi = (r-4, 1, 0, \dots, 0, -1, -(t-5) \mid 2, 1, 0, \dots, 0, -1, -3)$ , then  $\mu = ((r-4, 1), (2, 1))$  and  $\nu = ((t-5, 1), (3, 1))$  such that  $\xi = \mu - \hat{\nu}$ .

**Definition 6.2.** For any  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ , let  $\bar{\lambda} := \lambda_{pq} + \mu - \hat{\nu}$  and  $\tilde{\lambda} := \mu - \hat{\nu}$ . Since  $r + t \leq \min\{m, n\}$ , both  $\mu$  and  $\nu$  correspond to integral weights of  $\mathfrak{g}$  as above such that

$$\mu_i \nu_{m+1-i} = 0 \text{ for } 1 \leq i \leq m \text{ and } \mu_{m+j} \nu_{m+n+1-j} = 0 \text{ for } 1 \leq j \leq n, \quad (6.2)$$

**Lemma 6.3.** *For any  $\mathfrak{g}$ -highest weight  $\Lambda$  of  $M_{pq}^{rt}$ , there is a unique triple  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$  such that  $\Lambda = \bar{\lambda}$ .*

*Proof.* By [19, Lemma 5.20],  $\Lambda = \lambda_{pq} + \eta - \zeta$  for some bicompositions  $\eta$  and  $\zeta$  of sizes  $r$  and  $t$  respectively. For  $i \in I$ , let  $\xi_i = \min\{\eta_i, \zeta_i\}$  and  $f = \sum_{i \in I} \xi_i$ . Then we obtain a weight  $\xi$ , and two bicompositions  $\mu := \eta - \xi$  and  $\gamma := \zeta - \xi$  such that  $|\mu| = r - f$ ,  $|\gamma| = t - f$  and  $\Lambda = \lambda_{pq} + \mu - \gamma$ . Set  $\nu = \hat{\gamma}$ , then  $\Lambda = \bar{\lambda}$  and (6.2) is satisfied by definition of  $\xi$ . Since  $\Lambda$  is dominant,  $\mu, \nu$  must be bipartitions. Thus  $\Lambda$  corresponds to  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ . Such a  $\lambda$  is unique.  $\square$

**Definition 6.4.** For each  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ , denote  $v_\lambda = v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}}$ , where

$$\mathbf{i} = (\mathbf{i}_{\mu(1)}, \mathbf{i}_{\mu(2)}, \underbrace{1, \dots, 1}_f) \in I(m|n, r), \text{ and } \mathbf{j} = (\mathbf{j}_{\nu(2)}, \mathbf{j}_{\nu(1)}, \underbrace{1, \dots, 1}_f) \in \bar{I}(m|n, t),$$

such that

- (1)  $\mathbf{j}_{\nu(2)}$  is obtained from  $\mathbf{i}_{\nu(2)}$  by using  $m + n - i + 1$  instead of  $i$  for  $1 \leq i \leq n$ ,
- (2)  $\mathbf{j}_{\nu(1)}$  is obtained from  $\mathbf{i}_{\nu(1)}$  by using  $m - i + 1$  instead of  $i$  for  $1 \leq i \leq m$ .

For instance, if  $\lambda = (1, \mu, \nu) \in \Lambda_{2,8,10}$  with  $\mu = ((3, 1), (2, 1))$  and  $\nu = ((4, 1), (3, 1))$ , then  $\mathbf{i} = (1^3, 2, (m+1)^2, (m+2), 1)$  and  $\mathbf{j} = ((m+n)^3, (m+n-1), m^4, (m-1), 1)$ . Thus,

$$v_\lambda = v_1 \otimes v_{m+2} \otimes v_{m+1}^{\otimes 2} \otimes v_2 \otimes v_1^{\otimes 3} \otimes v_{pq} \otimes \bar{v}_{m+n}^{\otimes 3} \otimes \bar{v}_{m+n-1} \otimes \bar{v}_m^{\otimes 4} \otimes \bar{v}_{m-1} \otimes \bar{v}_1.$$

**Definition 6.5.** For any  $(f, \mu, \nu) \in \Lambda_{2,r,t}$ , define

- (1)  $w_{\mu, \nu} = w_\mu w_{\nu^\circ}$ , with  $\nu^\circ = (\nu^{(2)}, \nu^{(1)})$ ,  $w_\mu = d(\mathbf{t}_\mu) \in \mathfrak{S}_{r-f}$  and  $w_{\nu^\circ} = d(\mathbf{t}_{\nu^\circ}) \in \tilde{\mathfrak{S}}_{t-f}$ ,
- (2)  $v_{\lambda, \mathbf{t}, d, \kappa_d} = v_\lambda \mathbf{e}^f w_{\mu, \nu} \bar{\eta}_{\mu'} \bar{\eta}_{(\nu^\circ)'} d(\mathbf{t}) dx^{\kappa_d}$ ,  $\mathbf{t} \in \mathcal{T}^s(\mu') \times \mathcal{T}^s((\nu^\circ)'), d \in \mathcal{D}_{r,t}^f$  and  $\kappa_d \in \mathbf{N}_f$ .

**Theorem 6.6.** *Suppose  $r + t \leq \min\{m, n\}$ .*

- (1) *There is a bijection between the set of dominant weights of  $M_{pq}^{rt}$  and  $\Lambda_{2,r,t}$ .*
- (2) *If  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ , then  $V_{\bar{\lambda}}$ , the  $\mathbb{C}$ -space of all  $\mathfrak{g}$ -highest weight vectors of  $M_{pq}^{rt}$  with highest weight  $\bar{\lambda}$ , has a basis  $S := \{v_{\lambda, \mathbf{t}, d, \kappa_d} \mid \mathbf{t} \in \mathcal{T}^s(\mu') \times \mathcal{T}^s((\nu^\circ)'), d \in \mathcal{D}_{r,t}^f, \kappa_d \in \mathbf{N}_f\}$ .*

*Proof.* Obviously, (1) follows from Lemma 6.2. To obtain (2), we prove that for each  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ ,  $V_{\bar{\lambda}}$  has the required basis in the case either  $f = 0$  or  $f > 0$ .

*Case 1:  $f = 0$ .*

By Definition 6.5,  $v_{\lambda, \mathbf{t}, d, \kappa_d} = v_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}} w_\mu \bar{\eta}_{\mu'} d(\mathbf{t}_1) w_{\nu^\circ} \bar{\eta}_{(\nu^\circ)'} d(\mathbf{t}_2)$ , where  $\mathbf{i}, \mathbf{j}$  are defined in Definition 6.4. By Theorem 5.7,  $v_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}} w_\mu \bar{\eta}_{\mu'} d(\mathbf{t}_1)$  can be regarded as a  $\mathfrak{g}$ -highest weight vector of  $M_{pq}^{r0}$ . Similarly,  $v_{\mathbf{i}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}} w_{\nu^\circ} \bar{\eta}_{(\nu^\circ)'} d(\mathbf{t}_2)$  can be regarded as a  $\mathfrak{g}$ -highest weight vector of  $M_{pq}^{0t}$ . Thus,  $v_{\lambda, \mathbf{t}, d, \kappa_d}$  is a  $\mathfrak{g}$ -highest weight vector of  $M_{pq}^{rt}$ . The last assertion follows from arguments on counting the dimensions of  $V_{\bar{\lambda}}$  and that of  $\mathfrak{g}_0$ -highest weight vectors of  $V^{rt} := V^{\otimes r} \otimes W^{\otimes t}$  with highest weight  $\mu - \hat{\nu}$ .

*Case 2:  $f > 0$ .*

For any  $i \in I$ ,

$$v_i \otimes \bar{v}_i e_1 = (-1)^{[i]} \sum_{j \in I} v_j \otimes \bar{v}_j.$$



Thus  $v_i \otimes \bar{v}_i e_1$  is unique up to a sign for different  $i$ 's. Since  $M_{pq}^{rt}$  is a  $(\mathfrak{g}, \mathcal{B}_{2,r,t})$ -bimodule, we can switch  $v_{i_{r-k}}$  and  $\bar{v}_{j_{t-k}}$  in  $v_\lambda$  with  $i_{r-k} = j_{t-k}$  to  $v_o$  and  $\bar{v}_o$  for any fixed  $o, 1 \leq o \leq m+n$  simultaneously when we consider the action of  $E_{j,\ell}$  on  $i_{r-k}$ -th (resp.  $j_{t-k}$ -th) tensor factor of  $v_{\lambda,t,d,\kappa_d}$  for  $0 \leq k \leq f-1$ . Let

$$v_t := v_{i_{r-f}} \otimes \cdots v_{i_1} \otimes v_{pq} \otimes \bar{v}_{j_1} \otimes \cdots \bar{v}_{j_{t-f}} w_{\mu,\nu} x_{\alpha(2)} y_{\alpha(1)} y_{\beta(1)} x_{\beta(2)} \pi_{r-f-a}(-p) \pi_b(q) d(t), \quad (6.3)$$

where  $\alpha^{(i)}$  (resp.  $\beta^{(i)}$ ) is the conjugate of  $\mu^{(i)}$  (resp.  $\nu^{(i)}$ ),  $i = 1, 2$ . Applying Theorem 5.7 to both  $V^{\otimes r-f} \otimes K_{\lambda_{pq}}$  and  $K_{\lambda_{pq}} \otimes W^{\otimes t-f}$  yields  $E_{j,\ell} v_t = 0$ . So,  $E_{j,\ell} v_{\lambda,t,d,\kappa_d} = 0$  for any  $j < \ell$ .

We claim that  $S$  is linear independent, where  $S$  is given in (2). If so, each  $v_{\lambda,t,d,\kappa_d} \neq 0$ , forcing  $v_{\lambda,t,d,\kappa_d}$  to be a  $\mathfrak{g}$ -highest weight vector of  $M_{pq}^{rt}$  with highest weight  $\bar{\lambda}$ .

Suppose  $\mathbf{i} \in I(m|n, r_1-1)$  and  $\mathbf{j} \in \bar{I}(m|n, t_1-1)$  with  $r_1 \leq r$  and  $t_1 \leq t$  such that there are at least some  $k_0 \in I_0$  and  $\ell_0 \in I_1$  satisfying  $k_0, \ell_0 \notin \{i_l, j_o\}$  for all possible  $i, o$ 's. We consider  $\sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \bar{v}_k \in M_{pq}^{r_1, t_1}$ , where  $v \in B$  is a basis element of  $K_{\lambda_{pq}}$  in (4.6). Since  $x'_{r_1} = x_{r_1} + L_{r_1}$  and  $x'_{r_1}$  acts on  $M_{pq}^{r_1, t_1}$  as  $-\pi_{r_1,0}(\Omega)$ , where  $\Omega$  is given in (4.10), we have

$$\begin{aligned} \sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \bar{v}_k (x_{r_1} + L_{r_1}) &= -\pi_{r_1,0}(\Omega) \sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \bar{v}_k \\ &= - \sum_{k, i \in I} (-1)^{[k]+([k]+[i])([k]+[i])} v_i \otimes v_{\mathbf{i}} \otimes E_{k,i} v \otimes \bar{v}_j \otimes \bar{v}_k, \end{aligned}$$

where  $[i] = \sum_{j=1}^{r_1-1} [i_j]$ . So, up to some scalar  $a$ ,  $\sum_{k=1}^{m+n} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \bar{v}_k x_{r_1}$  contains the unique term  $v_{k_0} \otimes v_{\mathbf{i}} \otimes v \otimes \bar{v}_j \otimes \bar{v}_{k_0}$ . In particular, if  $v \neq v_{pq}$ ,  $\sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \bar{v}_k x_{r_1}$  does not contribute terms with form  $v_{k_0} \otimes v_{\mathbf{i}'} \otimes v_{pq} \otimes v_{\mathbf{j}'} \otimes \bar{v}_{k_0}$  for all possible  $\mathbf{i}'$  and  $\mathbf{j}'$ . If  $v = v_{pq}$ , by Lemma 5.4, the previous scalar is  $-p$ . Similarly, the coefficient of  $v_{\ell_0} \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \bar{v}_{\ell_0}$  in the expression of  $\sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v \otimes v_{\mathbf{j}} \otimes \bar{v}_k x_{r_1}$  is  $-q$ . Assume

$$c \sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \bar{v}_k x_{r_1} + d \sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \bar{v}_k = 0 \quad (6.4)$$

for some  $c, d \in \mathbb{C}$ . Then  $d = cp = cq$  by considering the coefficients of  $v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \bar{v}_k$ ,  $k \in \{k_0, \ell_0\}$  in the expression of LHS of (6.4). If  $c \neq 0$ , then  $p - q = 0$ . This is a contradiction since  $\lambda_{pq}$  is typical in the sense of (4.4). So,  $c = d = 0$  and hence  $\sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \bar{v}_k x_{r_1}$  and  $\sum_{k \in I} v_k \otimes v_{\mathbf{i}} \otimes v_{pq} \otimes v_{\mathbf{j}} \otimes \bar{v}_k$  are linear independent. Now, we assume

$$\sum_{t,d,\kappa_d} r_{t,d,\kappa_d} v_{\lambda,t,d,\kappa_d} = 0 \text{ for some } r_{t,d,\kappa_d} \in \mathbb{C}. \quad (6.5)$$

We claim that  $r_{t,d,\kappa_d} = 0$  for all possible  $t, d, \kappa_d$ . If not, then we pick up a  $d \in \mathcal{D}_{r,t}^f$  such that

- (1)  $r_{t,d,\kappa_d} \neq 0$ ,
- (2)  $d = s_{r-f+1, i_{r-f+1}} \bar{s}_{t-f+1, j_{t-f+1}} \cdots s_{r, i_r} \bar{s}_{t, j_t}$  and  $i_r > i_{r-1} > \cdots > i_{r-f+1}$ ,
- (3)  $(i_r, \dots, i_{r-f+1})$  is maximal with respect to lexicographic order.

Since  $r+t \leq \min\{m, n\}$  and  $0 < f \leq \min\{r, t\}$ , we can pick  $f$  pairs  $(k_i, \ell_i)$ ,  $r-f+1 \leq i \leq r$  such that

- (1)  $k_i \in I_0, \ell_i \in I_1, k_i > k_j$  and  $\ell_i > \ell_j$  if  $i > j$ ;
- (2) both  $v_{k_i}$  and  $v_{\ell_i}$  are not a tensor factor of  $v_{\mathbf{i}_\mu}$ ,
- (3) both  $\bar{v}_{k_i}$  and  $\bar{v}_{\ell_i}$  are not a tensor factor of  $\bar{v}_{\mathbf{j}}$ .

We consider the terms  $v_{\mathbf{a}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{b}}$ 's in the expressions of  $v_{\lambda, \mathbf{t}, d, \kappa_d}$ 's in LHS of (6.5) with  $r_{\mathbf{t}, d, \kappa_d} \neq 0$  such that either  $v_{a_{i_h}} = v_{k_h}$  and  $\bar{v}_{b_{i_t-r+h}} = \bar{v}_{k_h}$  or  $v_{a_{i_h}} = v_{\ell_h}$  and  $\bar{v}_{b_{i_t-r+h}} = \bar{v}_{\ell_h}$  for  $r-f+1 \leq h \leq r$ . Such terms occur in the expression of  $v_1^{\otimes f} \otimes \tilde{v}_{\mathbf{t}} \otimes \bar{v}_1^{\otimes f} \epsilon^f dx^{\kappa_d}$ , where  $\tilde{v}_{\mathbf{t}}$  is a linear combination of the terms in  $v_{\mathbf{t}}$ 's (cf. (6.3)) with forms  $v_{\mathbf{i}'} \otimes v_{pq} \otimes \bar{v}_{\mathbf{j}'}$ . If  $v_{a_h} = v_{k_h}$  and  $\bar{v}_{b_{i_t-r+h}} = \bar{v}_{k_h}$ , by previous arguments, the coefficient of  $v_{\mathbf{a}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{b}}$  in  $v_1^{\otimes f} \otimes v_{\mathbf{t}} \otimes \bar{v}_1^{\otimes f} \epsilon^f dx^{\kappa_d}$  is  $\prod_{h=r}^{r-f+1} (-p)^{\epsilon_h}$ , where  $\epsilon_h = 1$  if  $\kappa_h = 1$  and 0 if  $\kappa_h = 0$ . If  $v_{a_h} = v_{\ell_h}$  and  $\bar{v}_{b_{i_t-r+h}} = \bar{v}_{\ell_h}$ , then the coefficient of  $v_{\mathbf{a}} \otimes v_{pq} \otimes \bar{v}_{\mathbf{b}}$  in  $v_1^{\otimes f} \otimes \tilde{v}_{\mathbf{t}} \otimes \bar{v}_1^{\otimes f} \epsilon^f dx^{\kappa_d}$  is  $\prod_{h=r}^{r-f+1} (-q)^{\epsilon_h}$ , where  $\epsilon_h = 1$  if  $\kappa_h = 1$  and 0 if  $\kappa_h = 0$ . By (6.5),  $\sum_{\mathbf{t}} r_{\mathbf{t}, d, \kappa_d} \tilde{v}_{\mathbf{t}} = 0$  for any fixed  $\kappa_d$ . Thus, we can assume that  $\kappa_d = (0, \dots, 0) \in \mathbf{N}_f$ . If we identify  $\tilde{v}_{\mathbf{t}}$  with its  $v_{pq}$ -component, then  $\tilde{v}_{\mathbf{t}}$  can be considered as  $\mathfrak{g}_0$ -highest weight vectors of  $V^{\otimes r-f} \otimes W^{\otimes t-f}$  (cf. arguments in the proof of Theorem 5.7) of the form

$$\tilde{v}_{\mathbf{t}} = v_{i_{r-f}} \otimes \dots \otimes v_{i_1} \otimes \bar{v}_{j_1} \otimes \dots \otimes \bar{v}_{j_{t-f}} w_{\mu, \nu} x_{\alpha(2)} y_{\alpha(1)} \bar{y}_{\beta(1)} \bar{x}_{\beta(2)} d(\mathbf{t}).$$

So,  $r_{\mathbf{t}, d, \kappa_d} = 0$ , a contradiction. This proves that  $S$  is  $\mathbb{C}$ -linear independent. Further,  $S$  is a basis of  $V_{\bar{\lambda}}$  since the cardinality of  $S$  is  $2^f |\mathcal{D}_{r, \mathbf{t}}^f| \cdot |\mathcal{T}^s(\mu')| \cdot |\mathcal{T}^s(\nu')|$ , which is the dimension of space consisting of  $\mathfrak{g}_0$ -highest weight vectors of  $V^{rt}$  with highest weight  $\mu - \hat{\nu}$ .  $\square$

**Definition 6.7.** Let  $\mathfrak{F} = \text{Hom}_{U(\mathfrak{g})}(\cdot, M_{pq}^{rt})$  be the functor from the category of finite dimensional left  $\mathfrak{g}$ -modules to the category of right  $\mathcal{B}_{2, r, t}$ -modules over  $\mathbb{C}$ .

**Lemma 6.8.** *The functor  $\mathfrak{F}$  is exact.*

*Proof.* Since  $\lambda_{pq}$  is typical,  $M^{rt}$  is projective, injective and tilting as left  $\mathfrak{g}$ -module (e.g., [5, IV]). So,  $\mathfrak{F}$  is exact.  $\square$

**Proposition 6.9.** *Suppose  $\lambda = (f, \mu, \nu) \in \Lambda_{2, r, t}$ . Then  $\mathfrak{F}(K_{\bar{\lambda}}) \cong C(f, \mu', (\nu^o)')$ , where  $\nu^o = (\nu^{(2)}, \nu^{(1)})$ .*

*Proof.* By Proposition 3.9, there is an explicit linear isomorphism between  $C(f, \mu', (\nu^o)')$  and  $V_{\bar{\lambda}}$ , where  $V_{\bar{\lambda}}$  is given in Theorem 6.6. By Proposition 5.8 and [19, Proposition 6.10], this linear isomorphism is a  $\mathcal{B}_{2, r, t}$ -homomorphism. Thus,  $C(f, \mu', (\nu^o)') \cong V_{\bar{\lambda}}$  as right  $\mathcal{B}_{2, r, t}$ -modules. Using the universal property of Kac-modules yields  $\text{Hom}_{U(\mathfrak{g})}(K_{\bar{\lambda}}, M_{pq}^{rt}) \cong V_{\bar{\lambda}}$  as  $\mathcal{B}_{2, r, t}$ -modules (cf. the proof of Corollary 5.9). Now, everything is clear.  $\square$

In the remaining part of this section, we calculate decomposition matrices of  $\mathcal{B}_{2, r, t}$ . We always assume that  $p \in \mathbb{Z}$ . Otherwise, one can use  $x_1 + p_1$  instead of  $x_1$  for any  $p_1 \in \mathbb{C}$  with  $p - p_1 \in \mathbb{Z}$ . Since  $\lambda$  is typical, we have  $p - q \notin \mathbb{Z}$  or  $p - q \leq -m$  or  $p - q \geq n$ . In the first case, by [19, Theorem 5.21],  $\mathcal{B}_{2, r, t}$  is semisimple and hence its decomposition matrix is the identity matrix. We assume that  $p - q \leq -m$ . If  $p - q \geq n$ , one can switch the roles between  $p$  and  $q$  (or by considering the dual module of  $M_{pq}^{rt}$ ) in the following arguments.

Suppose  $\lambda = (f, \mu, \nu) \in \Lambda_{2, r, t}$ . Let  $T_{\bar{\lambda}}$  be the indecomposable tilting module, where  $\bar{\lambda} = \lambda_{pq} + \bar{\lambda} = \lambda_{pq} + \mu - \hat{\nu}$  (cf. Definition 6.2). It is the projective cover of  $L_{\bar{\lambda}}$ , where  $L_{\bar{\lambda}}$  is the simple  $\mathfrak{g}$ -module with highest weight  $\bar{\lambda}$ . It is known that  $T_{\bar{\lambda}}$  has filtrations of Kac-modules. Let  $K_{\bar{\lambda}_{\text{top}}}$  be the unique bottom of  $T_{\bar{\lambda}}$ . Then  $L_{\bar{\lambda}}$  is the simple  $\mathfrak{g}$ -module of  $K_{\bar{\lambda}_{\text{top}}}$ .

Further,  $\bar{\lambda}^{\text{top}}$  is the dominant weight defined in Definition 5.11 (1). Since  $M_{pq}^{rt}$  is a tilting module, it can be decomposed into the direct sum of indecomposable tilting modules

$$M_{pq}^{rt} = \bigoplus_{\mu \in P^+} T_{\mu}^{\oplus \ell_{\mu}} \quad \text{for some } \ell_{\mu} \in \mathbb{N}. \quad (6.6)$$

In the remaining part of this paper, we denote  $S$  to be the following finite subset of  $P^+$ ,

$$S := \{\mu \in P^+ \mid \ell_{\mu} \neq 0\}. \quad (6.7)$$

Parallel to Corollary 5.13, we have the following.

**Lemma 6.10.** *Let  $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$  such that  $(\lambda^{\text{top}})'$  is Kleshchev, where  $\lambda^{\text{top}}$  is defined in Definition 5.11 (2). Then  $T_{\bar{\lambda}}$  is a direct summand of  $M_{pq}^{rt}$ .*

*Proof.* We claim that  $T_{\bar{\lambda}}$  is a direct summand in  $M_{pq}^{r-f, t-f}$ . If so, then

$$v_1^f \otimes T_{\bar{\lambda}} \otimes \bar{v}_1^f \mathfrak{e}^f$$

is obviously a tilting submodule in  $M_{pq}^{rt}$  which is isomorphic to  $T_{\bar{\lambda}}$ . Thus the claim implies the result. Therefore, it suffices to consider the case  $f = 0$ .

Denote  $\bar{\nu} = \lambda_{pq} - \hat{\nu}$ . Since  $p \leq q - m$ , the weight diagram  $D_{\bar{\nu}}$  (cf. Definition 5.10) of  $\bar{\nu}$  is obtained from that of  $\lambda_{pq}$  in (5.12) by moving the “ $>$ ” at vertex  $p - i + 1$  to its left side at vertex  $p - i + 1 - \nu_{m-i+1}^{(1)}$  for each  $i$  with  $1 \leq i \leq m$ , and moving the “ $<$ ” at vertex  $q - m + j$  to its right side at vertex  $q - m + j + \nu_{n-j+1}^{(2)}$  for each  $j$  with  $1 \leq j \leq n$  (cf. (6.1)). Thus no “ $\times$ ” can be produced, i.e.,  $\bar{\nu}$  is typical. Hence  $K_{\bar{\nu}}$  is a direct summand in  $M_{pq}^{0t}$ . Thus, it suffices to prove that  $T_{\bar{\lambda}}$  is a direct summand in  $V^{\otimes r} \otimes K_{\bar{\nu}} \ltimes M_{pq}^{rt}$ , here  $\ltimes$  means direct summand of  $M_{pq}^{rt}$ . For this, we can apply [5, IV, Lemmas 2.4 and 2.6]. Note from [5, IV, Lemma 2.4] that the action of the functor  $F_i$  on  $K_{\bar{\nu}}$  defined in [5, IV] only depends on symbols at vertices  $i$  and  $i + 1$  of the weight diagram  $D_{\bar{\nu}}$  of  $\bar{\nu}$  (we remark that symbols  $\circ, \wedge, \vee, \times$  in [5, IV] are respectively symbols  $<, \times, \emptyset, >$  in this paper). Due to condition (6.2), for any  $i \in I_{pq} := I_{pq}^+ \setminus \{q - m + n\}$  such that  $i$  is involved in a path in the crystal graph in [5, IV, Lemma 2.6], the symbols at vertex  $i$  and  $i + 1$  in the weight diagram  $D_{\bar{\nu}}$  of  $\bar{\nu}$  are the same as that in the weight diagram  $D_{\emptyset}$  of  $\lambda_{pq}$ . This shows that  $T_{\bar{\lambda}}$  is a direct summand in  $V^{\otimes r} \otimes K_{\bar{\nu}}$  if and only if  $T_{\lambda_{pq} + \bar{\mu}}$  is a direct summand in  $V^{\otimes r} \otimes K_{\lambda_{pq}}$ , more precisely, [5, IV, Lemma 2.6] implies

$$F_{i_r} \cdots F_{i_1} K_{\bar{\nu}} \cong T_{\bar{\lambda}}^{\otimes 2\ell} \iff F_{i_r} \cdots F_{i_1} K_{\lambda_{pq}} \cong T_{\lambda_{pq} + \bar{\mu}}^{\otimes 2\ell},$$

where  $\ell$  is the number of edges in the given path of the form  $\emptyset \times \rightarrow < >$ . Thus the result follows from Corollary 5.13.  $\square$

We remark that there is a bijection between  $S$  defined in (6.7) and the set of pair-wise non-isomorphic simple  $\mathcal{B}_{2,r,t}$ -modules. See [19, Theorem 7.5]. For any  $\xi \in S$  as above, parallel to Definition 5.11, we define  $\xi^{\text{top}}$  to be the unique dominant weight such that  $L_{\xi}$  is the simple submodule of  $K_{\xi^{\text{top}}}$ .

**Proposition 6.11.** *For any  $\xi \in S$ , there is a unique  $(f, \mu, \nu) \in \Lambda_{2,r,t}$  such that  $\xi^{\text{top}} = \lambda_{pq} + \mu - \hat{\nu}$ . Further,  $\mathfrak{F}(T_{\xi})$  is isomorphic to the projective cover of  $D^{f, \mu', (\nu^0)'}$ , where  $D^{f, \mu', (\nu^0)'}$  is the simple head of  $C(f, \mu', (\nu^0)')$ .*

*Proof.* If  $\xi \in S$ , then  $T_\xi$  is an indecomposable tilting module with  $\ell_\xi > 0$ . By Theorem 4.4,  $\mathfrak{F}(T_\xi)$  is a direct sum of certain principle indecomposable right  $\mathcal{B}_{2,r,t}$ -modules. We claim that  $\mathfrak{F}(T_\xi)$  is indecomposable for any  $\xi \in S$ . Otherwise,  $\sum_{\xi \in S} \ell_\xi$  is strictly less than the number of principal indecomposable direct summands of right  $\mathcal{B}_{2,r,t}$ -module  $\mathcal{B}_{2,r,t}$ . However, for each principal indecomposable direct summand  $P$  of left  $\mathcal{B}_{2,r,t}$ -module  $\mathcal{B}_{2,r,t}$ ,  $P$  has to be a projective cover of irreducible left  $\mathcal{B}_{2,r,t}$ -module, say  $D$ , which is the simple head of a left cell module, say  $\Delta(\ell, \alpha, \beta)$  for some  $(\ell, \alpha, \beta) \in \Lambda_{2,r,t}$ , where  $\Delta(\ell, \alpha, \beta)$  is defined via a weakly cellular basis of  $\mathcal{B}_{2,r,t}$ . So, there is an epimorphism from  $P$  to  $\Delta(\ell, \alpha, \beta)$ . Since  $\mathfrak{G} := M_{pq}^{rt} \otimes_{\mathcal{B}_{2,r,t}} ?$  is right exact, there is an epimorphism from  $\mathfrak{G}(P)$  to  $\mathfrak{G}(\Delta(\ell, \alpha, \beta))$ . If  $\mathfrak{G}(\Delta(\ell, \alpha, \beta)) \neq 0$ , then  $\mathfrak{G}(P)$  is a non-zero direct summand of  $M_{pq}^{rt}$ . This implies that the number of indecomposable direct summands of left  $\mathcal{B}_{2,r,t}$ -module  $\mathcal{B}_{2,r,t}$  is strictly less than  $\sum_{\xi \in S} \ell_\xi$ . This is a contradiction since the number of principal indecomposable direct summands of left  $\mathcal{B}_{2,r,t}$ -module  $\mathcal{B}_{2,r,t}$  is equal to that of right  $\mathcal{B}_{2,r,t}$ -module  $\mathcal{B}_{2,r,t}$ . So,  $\mathfrak{F}(T_\xi)$  is indecomposable. Since  $K_{\xi^{\text{top}}} \hookrightarrow T_\xi$ , we have  $\mathfrak{F}(T_\xi) \twoheadrightarrow \mathfrak{F}(K_{\xi^{\text{top}}})$ . By Proposition 6.9,  $\mathfrak{F}(K_{\xi^{\text{top}}}) \cong C(f, \mu', (\nu^o)')$ . Thus,  $C(f, \mu', (\nu^o)')$  has the simple head, denoted by  $D^{f, \mu', (\nu^o)'}$ , and hence  $\mathfrak{F}(T_\xi) = P^{f, \mu', (\nu^o)'}$ . Since  $\xi \in S$ , by Lemma 6.10, both  $\mu'$  and  $(\nu^o)'$  are Kleshchev in the sense of (3.15) with respect to  $-p, m - q$  and  $q, p - n$ .

It remains to prove  $\mathfrak{G}(\Delta(\ell, \alpha, \beta^o)) \neq 0$  for any  $\delta := (\ell, \alpha, \beta) \in \Lambda_{2,r,t}$ . By Theorem 6.6,  $V_{\bar{s}}$  contains a non-zero vector  $v := v_1^{\otimes \ell} \otimes v_i \otimes v_{pq} \otimes v_j \otimes \bar{v}_1^{\otimes \ell} \epsilon^f w_{\alpha, \beta} \eta_{\alpha'} \bar{\eta}_{(\beta^o)'}$ , where  $i$  and  $j$  are defined as in Definition 6.4. So, it is enough to show  $v \in \mathfrak{G}(\Delta(\ell, \alpha, \beta^o))$ , where  $\Delta(\ell, \alpha, \beta^o)$  is defined via a suitable weakly cellular basis of  $\mathcal{B}_{2,r,t}$ . We use cellular bases of  $\mathcal{H}_{2,r-f}$  and  $\mathcal{H}_{2,t-f}$  in Lemma 3.3 (1) (3) to construct a weakly cellular basis of  $\mathcal{B}_{2,r,t}$ , which is similar to that in Theorem 3.6. Let  $\Delta(\ell, \alpha, \beta^o)$  be the corresponding left cell module with respect to  $(\ell, \alpha, \beta^o) \in \Lambda_{2,r,t}$ . By arguments similar to those for the proof of Proposition 3.9, one can verify

$$\Delta(\ell, \alpha, \beta^o) \cong \mathcal{B}_{2,r,t} \epsilon^f \mathfrak{r}_{\alpha} \bar{\mathfrak{r}}_{\beta^o} w_{\alpha, \beta} \eta_{\alpha'} \bar{\eta}_{(\beta^o)'} \pmod{\mathcal{B}_{2,r,t}^{\ell+1}}.$$

Let  $M = \tilde{v} \mathcal{B}_{2,r,t}$  be the cyclic  $\mathcal{B}_{2,r,t}$ -module generated by  $\tilde{v} := v_1^{\otimes \ell} \otimes v_i \otimes v_{pq} \otimes v_j \otimes \bar{v}_1^{\otimes \ell}$ . Then  $M \otimes_{\mathcal{B}_{2,r,t}} \Delta(\ell, \alpha, \beta^o)$  is a subspace of  $\mathfrak{G}(\Delta(\ell, \alpha, \beta^o))$ . Since  $\mathcal{B}_{2,r,t}^{\ell+1}$  acts on  $M$  trivially, there is a  $\mathbb{C}$ -linear map  $\phi : M \otimes_{\mathcal{B}_{2,r,t}} \Delta(\ell, \alpha, \beta^o) \rightarrow M$  such that  $\phi(m \otimes \bar{h}) = mh$  for any  $\bar{h} \in \mathcal{B}_{2,r,t} \epsilon^f \mathfrak{r}_{\alpha} \bar{\mathfrak{r}}_{\beta^o} w_{\alpha, \beta} \eta_{\alpha'} \bar{\eta}_{(\beta^o)'} \pmod{\mathcal{B}_{2,r,t}^{\ell+1}}$ . Since  $\lambda_{pq}$  is typical and the ground field is  $\mathbb{C}$ , up to a non-zero scalar, we have  $v = \phi(\tilde{v} \otimes \bar{h})$ , where  $h \equiv \epsilon^f w_{\alpha, \beta} \eta_{\alpha'} \bar{\eta}_{(\beta^o)'} \pmod{\mathcal{B}_{2,r,t}^{\ell+1}}$ . Thus,  $\mathfrak{G}(\Delta(\ell, \alpha, \beta^o)) \neq 0$ .  $\square$

*Remark 6.12.* Proposition 6.11 implies that  $C(f, \mu, \nu)$  has the simple head if  $\mu$  and  $\nu$  are Kleshchev bipartitions with respect to  $-p, m - q$  and  $q, p - n$  in the sense of (3.15). Further, all non-isomorphic simple  $\mathcal{B}_{2,r,t}$ -modules can be realized in this way.

**Proposition 6.13.** *Suppose  $\xi \in P^+$ . Then  $\mathfrak{F}(L_\xi) = 0$  if  $\xi \notin S$  (cf. (6.7)) and  $\mathfrak{F}(L_\xi) \cong D^{f, \mu', (\nu^o)'}$  if  $\xi \in S$ , where  $\xi^{\text{top}} = \lambda_{pq} + \mu - \hat{\nu}$  with  $(f, \mu, \nu) \in \Lambda_{2,r,t}$ .*

*Proof.* By (6.6),  $\mathfrak{F}(L_\xi) = \oplus_{\zeta \in S} \text{Hom}_{\mathfrak{g}}(L_\xi, T_\zeta^{\otimes \ell_\zeta})$ . Suppose  $0 \neq f \in \text{Hom}_{U(\mathfrak{g})}(L_\xi, T_\zeta^{\oplus \ell_\zeta})$ . Then  $L_\xi \cong f(L_\xi)$  is a simple submodule of  $T_\zeta^{\oplus \ell_\zeta}$ . Since  $T_\zeta$  has the unique simple submodule  $L_\zeta$ ,

$\mathfrak{F}(L_\xi) = 0$  if  $\xi \notin S$ . If  $\xi \in S$ , then

$$\mathfrak{F}(L_\xi) = \text{Hom}_{U(\mathfrak{g})}(L_\xi, T_\xi^{\oplus \ell_\xi}), \quad (6.8)$$

which is obviously  $\ell_\xi$ -dimensional. Let  $v_\xi^1, \dots, v_\xi^{\ell_\xi} \in T_\xi^{\oplus \ell_\xi}$  be the generators of the tilting module  $T_\xi^{\oplus \ell_\xi}$  (then  $v_\xi^1, \dots, v_\xi^{\ell_\xi}$  span the generating space, denoted  $\mathbf{V}$ , of  $T_\xi^{\oplus \ell_\xi}$ ), and  $v_\xi'^1, \dots, v_\xi'^{\ell_\xi} \in L_\xi^{\oplus \ell_\xi}$ , the corresponding generators of the submodule  $L_\xi^{\oplus \ell_\xi}$  of  $T_\xi^{\oplus \ell_\xi}$ . Thus, there exists a unique  $u \in U(\mathfrak{g})$  such that

$$v_\xi'^i = uv_\xi^i \text{ for } i = 1, \dots, \ell_\xi. \quad (6.9)$$

Let  $\tilde{v}_\xi \in L_\xi$  be the generator of the simple module  $L_\xi$ . As in the proof of Corollary 5.14, we can define  $f^i : L_\xi \rightarrow T_\xi^{\oplus \ell_\xi}$  to be the  $U(\mathfrak{g})$ -homomorphism sending  $\tilde{v}_\xi$  to  $v_\xi'^i$  for  $i = 1, \dots, \ell_\xi$ . Then  $(f^1, \dots, f^{\ell_\xi})$  is obviously a basis of  $\mathfrak{F}(L_\xi)$  (cf. (6.8)).

For any  $A \in M_{\ell_\xi}$  (the algebra of  $\ell_\xi \times \ell_\xi$  complex matrices), we can define an element  $\phi_A \in \text{End}_{U(\mathfrak{g})}(M_{pq}^{rt}) = \mathcal{B}_{2,r,t}$  as follows:  $\phi_A|_{T_\zeta^{\oplus \ell_\zeta}} = 0$  if  $\zeta \neq \xi$  and

$$\phi_A|_{T_\xi^{\oplus \ell_\xi}} : (v_\xi^1, \dots, v_\xi^{\ell_\xi}) \mapsto (v_\xi^1, \dots, v_\xi^{\ell_\xi})A \text{ (regarded as vector-matrix multiplication)}, \quad (6.10)$$

i.e., the transition matrix of the action of  $\phi_A|_{T_\xi^{\oplus \ell_\xi}}$  on the generating space  $\mathbf{V}$  of  $T_\xi^{\oplus \ell_\xi}$  under the basis  $(v_\xi^1, \dots, v_\xi^{\ell_\xi})$  is  $A$ . By the universal property of projective modules, this uniquely defines an element  $\phi_A \in \mathcal{B}_{2,r,t}$ . Thus we have the embedding  $\phi : M_{\ell_\xi} \rightarrow \mathcal{B}_{2,r,t}$  sending  $A$  to  $\phi_A$ . Write  $A$  as  $A = (a_{ij})_{i,j=1}^{\ell_\xi}$ . Then by (6.10) and definition of the right action of  $\mathcal{B}_{2,r,t}$  on  $M_{pq}^{rt}$ , we have

$$f^i(\tilde{v}_\xi)\phi_A = v_\xi'^i\phi_A = (uv_\xi^i)\phi_A = u(v_\xi^i\phi_A) = u \sum_{j=1}^{\ell_\xi} a_{ji}v_\xi^j = \sum_{j=1}^{\ell_\xi} a_{ji}v_\xi'^j = \left( \sum_{j=1}^{\ell_\xi} a_{ji}f^j \right)(\tilde{v}_\xi), \quad (6.11)$$

i.e., the transition matrix of the action of  $\phi_A$  on  $\mathfrak{F}(L_\xi)$  under the basis  $(f^1, \dots, f^{\ell_\xi})$  is  $A$ . Thus  $\phi(M_{\ell_\xi})$  acts transitively on the  $\ell_\xi$ -dimensional space  $\mathfrak{F}(L_\xi)$  and hence  $\mathfrak{F}(L_\xi)$  is a simple  $\mathcal{B}_{2,r,t}$ -module. Finally, since  $L_\xi \hookrightarrow K_{\xi^{\text{top}}}$ , we have  $\mathfrak{F}(K_{\xi^{\text{top}}}) \twoheadrightarrow \mathfrak{F}(L_\xi)$ . Note that  $D^{f, \mu', (\nu^o)'} is the simple head of  $\mathfrak{F}(K_{\xi^{\text{top}}})$ . Thus,  $\mathfrak{F}(L_\xi) \cong D^{f, \mu', (\nu^o)'}$ .  $\square$$

**Theorem 6.14.** *Suppose  $(f, \alpha, \beta) \in \Lambda_{2,r,t}$  such that there is a  $\lambda \in S$  (cf. (6.7)) satisfying  $\lambda^{\text{top}} = \lambda_{pq} + \alpha - \hat{\beta}$ . If  $\mu := (\ell, \gamma, \delta) \in \Lambda_{2,r,t}$ , then  $[C(\ell, \gamma', (\delta^o)') : D^{f, \alpha', (\beta^o)'}] = (T_\lambda : K_{\bar{\mu}})$ .*

*Proof.* The result follows from Lemma 6.8, Propositions 6.9 and 6.13, together with the BGG reciprocity formula for  $\mathfrak{g}$ .  $\square$

## REFERENCES

- [1] S. ARIKI and A. MATHAS, “The number of simple modules of the Hecke algebras of type  $G(r, 1, n)$ ”, *Math. Z.* **233** (2000), 601–623.
- [2] S. ARIKI and A. MATHAS, “On the classification of simple modules for cyclotomic Hecke algebras of type  $G(m, 1, n)$  and Kleshchev multipartitions”, *Osaka J. Math.* **38** (2001), 827–837.
- [3] S. ARIKI, A. MATHAS and H. RUI, “Cyclotomic Nazarov-Wenzl algebras”, *Nagoya Math. J.*, Special issue in honor of Prof. G. Lusztig’s sixty birthday, **182** (2006), 47–134.
- [4] J. BRUNDAN, “Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra  $\mathfrak{gl}_{m|n}$ ”, *J. Amer. Math. Soc.* **16** (2002), 185–231.

- [5] J. BRUNDAN and C. STROPPEL, “Highest weight categories arising from Khovanov’s diagram algebra I, II, III, IV”, *Moscow Math. J.* **11**, (2011), 685–722; *Transform. Groups* **15**, (2010), 1–45; *Represent. Theory* **15**, (2011), 170–243; *J. Eur. Math. Soc.* **14**, (2012), 373–419.
- [6] J. BRUNDAN and C. STROPPEL, “Gradings on walled Brauer algebras and Khovanov’s arc algebra, *Adv. Math.* **231** (2012), no. 2, 709–773.
- [7] R. DIPPER and G. G. JAMES “Representations of Hecke algebras of type  $B_n$ ”, *J. Algebra* **146** (1992), 454–481.
- [8] R. DIPPER and G. JAMES AND E. MURPHY “Hecke algebras of type  $B_n$  at roots of unity”, *Proc. London Math. Soc.* (3) **70** (1995), no. 3, 505–528.
- [9] R. DIPPER and A. MATHAS, “Morita equivalences of Ariki–Koike algebras”, *Math. Z.* **240** (2002), 579–610.
- [10] V.G. DRINFELD, “Degenerate affine Hecke algebras and Yangians”, *Func. Anal. Appl.*, **20** (1986), 62–64.
- [11] F. GOODMAN and J. GRABER, “Cellularity and the Jones basic construction”, *Adv. in Appl. Math.* **46** (2011), 312–362
- [12] J. J. GRAHAM and G. I. LEHRER, “Cellular algebras”, *Invent. Math.* **123** (1996), 1–34.
- [13] C. GROSON and V. SERGANOVA, “Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras”, *Proc. London Math. Soc.* **101** (2010), 852–892.
- [14] V.G. KAC, “Lie superalgebras”, *Adv. Math.* **26** (1977), 8–96.
- [15] A. KLESHCHEV, “Linear and projective representations of symmetric groups”, Cambridge Tracts in Mathematics, 163. Cambridge University Press, Cambridge, 2005.
- [16] K. KOIKE, “On the decomposition of tensor products of the representations of classical groups: By means of universal characters”, *Adv. Math.* **74** (1989) 57–86.
- [17] P. NIKITIN, “The centralizer algebra of the diagonal action of the group  $GL_n(\mathbb{C})$  in a mixed tensor space”, *J. Math. Sci.* **141** (2007), 1479–1493.
- [18] H. RUI and L. SONG, “Decomposition numbers of quantized walled Brauer algebras”, preprint, 2011.
- [19] H. RUI and Y. SU, “Affine walled Brauer algebras and super Schur-Weyl duality”, arXiv:1305.0450.
- [20] A. SARTORI, “The degenerate affine walled Brauer algebra”, arXiv: 1305.2347
- [21] Y. SU, J.W.B. HUGHES and R.C. KING, “Primitive vectors in the Kac-module of the Lie superalgebra  $sl(m|n)$ ”, *J. Math. Phys.* **41** (2000), 5044–5087.
- [22] Y. SU and R.B. ZHANG, “Generalised Jantzen filtration of Lie superalgebras I”, *J. Eur. Math. Soc.* **14** (2012), 1103–1133.
- [23] V. TURAEV, “Operator invariants of tangles and R-matrices”, *Izv. Akad. Nauk SSSR Ser. Math.* **53** (1989) 1073–1107 (in Russian).
- [24] J. VAN DER JEUGT and R.B. ZHANG, “Characters and composition factor multiplicities for the Lie superalgebra  $gl(m|n)$ ”, *Lett. Math. Phys.* **47** (1999), 49–61.
- [25] M. VAZIRANI, “Parameterizing Hecke algebra modules: Bernstein-Zelevinsky multisegments, Kleshchev multipartitions, and crystal graphs”, *Transformation Groups* **7** (2002), no. 3, 267–303.

H. RUI: DEPARTMENT OF MATHEMATICS, SHANGHAI KEY LABORATORY OF PMMP, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, 200241, CHINA

*E-mail address:* hbrui@math.ecnu.edu.cn

Y. SU: DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 200092, CHINA

*E-mail address:* ycsu@tongji.edu.cn